New Convergence Techniques for Identifying with Fixed Point by Nonexpansive Mappings

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Abstract – This paper aimed to be generating new iterative outcomes and double step iterative techniques for recognizing of fixed points of nonexpansive mappings in Banach space. Encourage we demonstrate another iterative procedure, which is finer than different other existing iterative methods.

1. INTRODUCTION

We take a set E which is uniformly convex Banach space and E is the super set of C and C is closed convex set. In this paper, N indicates the set of all positive integers and \( G (T) = \{ x : Gx = x \} \). A mapping \( T : C \to C \) is called nonexpansive if for any arbitrary we can take \( x_n \in C \), Generate a sequence \( \{ x_n \} \), where \( x_n \) is characterized by positive integer \( n \geq 1 \) as:

\[
x_{n+1} = Tx_n, \quad (i)
\]

Hear (i) known as Picard sequence.

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad (ii)
\]

Here (ii) known as Mann [8] sequence.

\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTx_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n
\end{align*}
\]

(iii)

Here (iii) known as Ishikawa [5] sequence.

Ishikawa, Mann and other iteration methods have considered by few researchers for approximation fixed point of nonexpansive mapping [6, 11, 13-15].

Noor [9] defined iterative method by \( x_1 \in C \), as follows:

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n,
\]

\[
y_n = (1 - \beta_n)x_n + \beta_nTx_n
\]

\[
z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \quad (vi)
\]

Where \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) are sequence in (0, 1).

Agrawal [2] in 2007 constructs accompanying strategy:

\[
x_{n+1} = (1 - \alpha_n)Ty_n + \alpha_nTz_n,
\]

\[
y_n = (1 - \beta_n)x_n + \beta_nTz_n,
\]

\[
z_n = (1 - \gamma_n)x_n + \gamma_nTx_n \quad (v)
\]

Where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are in (0, 1). They demonstrated that this scheme converges at equivalence comparatively as Picard schemes and it’s comparatively finer than Mann iterative schemes for contraction mappings.

Abbas et. al. [1] constructs the other following schemes, where sequence \( \{ x_n \} \) is generated from any arbitrary \( x_1 \in C \)

\[
x_{n+1} = (1 - \alpha_n)Ty_n + \alpha_nTz_n,
\]

\[
y_n = (1 - \beta_n)x_n + \beta_nTz_n,
\]

\[
z_n = (1 - \gamma_n)x_n + \gamma_nTx_n \quad (vi)
\]
Where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are sequence in \((0, 1)\). They proved that this scheme converges as finer than iterative techniques (v) [2].

Now, motivated by above all techniques we build another iterative process for determining the fixed point of nonexpansive mapping. Where sequence \( \{x_n\} \) is generated by \( x_i \in C \) and written as follows:

\[
\begin{align*}
x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_nTy_n, \\
y_n &= (1-\beta_n)x_n + \beta_nTx_n,
\end{align*}
\]

(vii)

The primary goal of this article to satisfies convergent outcomes for nonexpansive mappings by iteration (vii). We also show that iteration (vii) converges faster than various other iterative schemes.

2. PRELIMINARIES:

Consider \( K \) is a Banach space and \( S_K = \{x \in K : \|x\| = 1\} \) is sphere on \( K \) having magnitude is identity. For all \( \delta \in (0,1) \) and \( x, y \in S_K \) with \( x \neq y \), if \( \| (1-\delta)x + \delta y \| < 1 \), then is said to be convex i.e. strictly convex Banach space. \( \|x\| = \|y\| = \|\alpha x + (1-\alpha)y\| \) and \( \alpha \in (0,1) \), then \( x = y \).

Then space \( K \) is called smooth if

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = 0.
\]

The space \( K \) is satisfies the Opial’s condition [10] for every sequence \( \{x_n\} \) in \( K \), i.e

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \quad \forall y \in K \text{ with } x \neq y.
\]

**Definition 2.1** Assume that \( \{r_n\} \) and \( \{s_n\} \) are sequences that both sequences have limit point i.e both are converges to \( r \) and \( s \). if

\[
\lim_{n \to \infty} \frac{|r_n - r|}{s_n - s} = 0,
\]

then \( \{r_n\} \) converges faster than \( \{s_n\} \).

**Definition 2.2** Consider two fixed point iteration processes \( \{p_n\} \) and \( \{q_n\} \), both are converging to common fixed point at \( t \), then the error estimates are as below:

\[
\|p_n - t\| \leq r_n \quad \text{and} \quad \|q_n - t\| \leq s_n \quad \text{both are defined for } n \geq 1,
\]

Here \( \{r_n\} \) and \( \{s_n\} \) are two real sequences of positive number tends to 0. If \( \{r_n\} \) converges finer than \( \{s_n\} \), then \( \{p_n\} \) is also finer than \( \{u_n\} \) to \( t \).

Here we will characterize a few lemmas for further uses in this paper which are as follows:

**Lemma 2.3** [4] Assume \( C \) is a nonempty closed convex Banach space \( K \) and \( T \) is a nonexpansive mapping on \( C \). So \( I - T \) is demiclosed at 0.

**Lemma 2.4**[12] Consider \( E \) is a uniformly convex Banach space and \( 0 < p \leq t_n \leq q < 1 \). Now \( \{x_n\} \) and \( \{y_n\} \) is the two sequences of \( E \) such that \( \limsup_{n \to \infty} \|x_n\| \leq r \), \( \limsup_{n \to \infty} \|y_n\| \leq r \) and

\[
\limsup_{n \to \infty} \left\| r_n x_n + (1-t_n) y_n \right\| = r, \text{ hold for some } r \geq 0. \text{ Then } \lim_{n \to \infty} \|x_n - y_n\| = 0.
\]

**Lemma 2.5** [2] Assume that \( E \) be a reflexive Banach space fulfilling the Opial condition and a function \( T: C \rightarrow X \) such that \( I - T \) is demiclosed at 0 and \( F(T) \neq \emptyset \) where \( C \) is a convex subset of \( E \). Let \( \{x_n\} \) be a sequence in \( C \) such that \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \) and \( \lim_{n \to \infty} \|x_n - p\| = 0 \) exist for all \( p \in F(T) \). Then \( \{x_n\} \) converges to a fixed point of \( T \).

3. CONVERGENCE RATE

In this part, we will prove that iteration method (vii) converges finer than the other methods.

**Theorem 3.1** Let we define a set \( E \) such that set \( E \) is normed linear space and there is the subset \( C \) of \( E \), which is nonvoid closed convex set and let \( T \) be a constructive mapping included a factor \( k \in (0,1) \) and fixed point \( p \). Let \( \{u_n\} \) be characterized by the iteration techniques (vi) and \( \{x_n\} \) by (vii), where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are in
[ε,1−ε] \forall n ∈ N and for some ε in (0, 1). So \{x_n\} converges finer from \{u_n\}.

**Proof:** As demonstrated in Theorem 3 of M. Abbas and T. Nazir[1].

\[ \|u_{n+1} - p\| \leq k^n \{1 - (1 - k)\alpha\beta\gamma^n\} \|u_1 - p\|, \quad n \in N. \]

Let \(a_n = k^n \{1 - (1 - k)\alpha\beta\gamma^n\}\|u_1 - p\|\]

\[ \|y_n - p\| = \|(1 - \beta_n)x_n + \beta_nTx_n - p\| \]

\[ \leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \]

\[ = (1 - (1 - k)\beta_n)\|x_n - p\| \]

Thus

\[ \|x_{n+1} - p\| = \|(1 - \alpha_n)Tx_n + \alpha_nTy_n - p\| \]

\[ \leq (1 - \alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\| \]

\[ \leq (1 - \alpha_n)\|Tx_n - p\| + \alpha_n(1 - \beta_n)\|x_n - p\| + \alpha_n\beta_n\|Tx_n - p\| \]

\[ \leq k\{1 - \alpha_n + \alpha_n(1 - (1 - k)\beta_n)\}\|x_n - p\| \]

\[ = k\{1 - (1 - k)\alpha\beta\gamma^n\}\|x_1 - p\| \]

Suppose \(b_n = k^n \{1 - (1 - k)\alpha\beta\gamma^n\}\|x_1 - p\|\]

\[ \frac{b_n}{\alpha_n} = \frac{k^n \{1 - (1 - k)\alpha\beta\gamma^n\}\|x_1 - p\|}{k^n \{1 - (1 - k)\alpha\beta\gamma^n\}\|x_1 - p\|} \rightarrow 0 \]

as \(n \rightarrow \infty.\)

Consequently \{x_n\} converges finer than \{u_n\}.

**Theorem 3.2** Here we consider a self mapping \(T\) on set \(C\) and set \(C\) is nonvoid closed set. Now assume \(E\) as a normed linear space \(E\) where \(E\) is the super set of \(C\), a sequence \{\(x_n\)\} defined by (vii) and \(F(T) \neq \emptyset\). Then \(\lim_{n \to \infty} \|x_n - p\|\) exist for each \(p \in F(T)\).

**Proof.** Now by (vii), we have

\[ \|y_n - p\| = \|(1 - \beta_n)x_n + \beta_nTx_n - p\| \]

\[ \leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \]

\[ \leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \]

\[ = \|x_n - p\| \]

Thus \(\lim_{n \to \infty} \|x_n - p\|\) exists \(\forall p \in F(T)\).

**Theorem 3.3** Let us have a set \(E\) which is uniformly Banach space and \(E\) is the super set of \(C\) i.e \(C \subseteq E\) where \(C\) is nonempty closed convex set.

Let \(T\) be a nonexpansive self-mapping on \(C\), and a sequence \{\(x_n\)\} provided by (vii) and \(F(T) \neq \emptyset\) then

\[ \lim_{n \to \infty} \|x_n - Tx_n\| = 0. \]

**Proof.** By theorem 3.2 we have

\[ \lim_{n \to \infty} \|x_n - p\| \]

is exists.

Then assume \(\lim_{n \to \infty} \|x_n - p\| = c\).

By from (viii) and (ix) we have

\[ \lim \sup_{n \to \infty} \|x_n - p\| \leq c, \]

and

\[ \lim \|x_n - p\| \leq c \]

as \(n \to \infty.\)

But \(T\) is a nonexpansive map so we have

\[ \|Tx_n - p\| \leq \|x_n - p\| \] and \[ \|Ty_n - p\| \leq \|y_n - p\| \]

After getting \(\limsup\) on both sides,
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\[ \limsup_{n \to \infty} \| Tx_n - p \| \leq c \quad (x) \]

And

\[ \limsup_{n \to \infty} \| Ty_n - p \| \leq c \quad (xii) \]

Since

\[ \lim_{n \to \infty} \| x_{n+1} - p \| = \lim_{n \to \infty} (1 - \alpha_n) (Tx_n - p) + \alpha_n (Ty_n - p) \]

Hence by the using of lemma 2.4

\[ \lim_{n \to \infty} \| x_n - y_n \| = 0. \]

Now form (x) and (xii)

\[ \lim_{n \to \infty} \| Tx_n - p \| = c, \]

additionally we have

\[ \| Ty_n - p \| \leq \| Ty_n - Tx_n \| + \| Tx_n - p \| \]

\[ \leq \| Ty_n - Tx_n \| + \| y_n - p \| \]

and

\[ \limsup_{n \to \infty} \| Ty_n - p \| \geq c \quad (xiv) \]

So from (xii) & (xiv) and

\[ \lim_{n \to \infty} \| y_n - p \| = c. \]

Now by utilizing the lemma 2.4, from we have

\[ \lim_{n \to \infty} \| y_n - Ty_n \| = 0. \]

since \( c = \lim_{n \to \infty} \| y_n - p \| \)

\[ = \lim_{n \to \infty} (1 - \beta_n) x_n + \beta_n T x_n - p \]

Hence the theorem is verified.

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