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Analytical properties and solution methods of differential equations involving Atangana–baleanu fractional operators

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Abstract: The fractional calculus is known as a powerful mathematical tool for the description of systems with memory, hereditary properties and non-locality. The Atangana–Baleanu (AB) fractional operators based on the non-singular, non-local Mittag-Leffler kernel has attracted much research attention of the fractional operators developed to overcome the singular-kernel limitations of the classical Riemann–Liouville and Caputo derivatives. In this paper, a systematic analytical study of differential equations with Atangana-Baleanu fractional operators are presented. We introduce an extended modified Atangana-Baleanu Caputo derivative together with its associated integral operator and prove their main properties including linearity, boundedness, non-locality and consistency with the classical derivatives. A generalised Laplace transform of the extended operator is derived and employed to develop methods for analytical solutions of linear and nonlinear Atangana-Baleanu fractional differential equations (AB-FDEs) including initial value and boundary value problems. Existence, uniqueness and Ulam-Hyers stability of solutions are proved via Banach and Krasnoselskii type fixed point theorems. Furthermore, within the extended AB framework we obtain new fractional derivative relations for the Gamma, Beta, hypergeometric and orthogonal polynomial functions. Numerical examples are provided to validate the theoretical results and a comparative discussion is provided to place the Atangana–Baleanu operator with regard to classical fractional derivatives. The results provide analytical tools that help in the treatment of memory dependent differential equations arising in applied mathematics, physics and engineering.

Keywords: Atangana–Baleanu fractional derivative, Mittag-Leffler kernel, fixed-point theorem, Laplace transform, Ulam–Hyers stability,Fractional calculus

1. INTRODUCTION

Fractional calculus has gained significance in recent years as an analytical tool for modeling complex dynamical systems with memory and inheritance. Whereas classical calculus limits differentiation and integration to integers, fractional calculus extends differentiation and integration to any real or complex order. The ability of a single operator to represent the whole history of a process is critical in modeling viscoelastic materials, anomalous diffusion, control systems, and other physical and engineering applications not adequately represented by integer-order models. The earliest and most frequently encountered fractional operators are the Riemann–Liouville (RL) and Caputo derivatives. However, both types are based on power-law or singular kernels with singularities at zero. This gives rise to analytical and computational problems, especially in modeling systems with smooth and gradually decreasing memory (Caputo, 1967). To remedy these problems, nonsingular kernel operators have been proposed. The Caputo–Fabrizio operator, which utilizes an exponential kernel, became available in 2015; shortly thereafter, Atangana and Baleanu introduced a fractional operator based on the Mittag-Leffler function — the Mittag-Leffler function is itself an appropriate generalization of the exponential function and can be closely associated with the solutions to fractional differential equations (Atangana &Baleanu, 2016).

The Atangana–Baleanu (AB) operator is available in RiemannLiouville (ABR) and Caputo (ABC) forms, and combines non-singularity and nonlocality such that it provides a more physically realistic representation of memory effects compared to either classical power-law operators or the Caputo–Fabrizio operator. Since its introduction, the AB operator has provided a framework for reformulating a wide variety of differential equations. The analytical properties of the AB operator, the integral transforms associated with the AB operator, and the methods of numerically implementing the AB operator have led to a large body of literature (Baleanu, Fernandez & Akgül, 2018). In spite of this progress, there are a number of analytical issues that require further investigation: boundedness and consistency properties of generalized AB-type operators, designing a Laplace transform that works with generalized AB-type operators, establishing general existence–uniqueness and stability theorems for the corresponding fractional differential equations, and creating constructive analytical solution methods applicable to both linear and nonlinear fractional differential equations are all issues that will be examined in the context of a unified analytical framework.

2. PRELIMINARIES

In this section we recall the definitions and basic results needed in the following analysis.

Gamma and Mittag-Leffler Functions

The Gamma function, defined for $Re(z) > 0$, generalizes the factorial and appears in almost every fractional operator through its role in normalizing power-law kernels:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

The one-parameter and two-parameter Mittag-Leffler functions are defined respectively as

$$E_\alpha(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + 1)\ ,\quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + \beta)$$

for $\alpha, \beta > 0$ and $z \in \mathbb{C}$. These functions reduce to the classical exponential function when $\alpha = \beta = 1$, and plays the same structural role in fractional calculus as the exponential function in classical calculus, especially as the kernel of the Atangana-Baleanu operators and the natural form of solution for linear fractional differential equations(Gorenflo, Kilbas, Mainardi & Rogosin, 2014).

The Atangana–Baleanu Fractional Derivative and Integral

For a function $f \in H^1(a,b)$ and order $\alpha \in (0,1)$, the Atangana–Baleanu derivative in the Caputo sense (ABC) is defined as

$$D^\alpha_{ABC}\, f(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(\tau)\, E_\alpha[-\alpha(t-\tau)^\alpha / (1-\alpha)]\,d\tau$$

where $B(\alpha)$ is a normalization function satisfying $B(0) = B(1) = 1$ (Atangana & Baleanu, 2016). The corresponding derivative in the Riemann-Liouville sense (ABR) is obtained by taking the derivative outside of the integral:

$$D^\alpha_{ABR}\, f(t) = \frac{B(\alpha)}{1-\alpha}\, d/dt \int_a^t f(\tau)\, E_\alpha[-\alpha(t-\tau)^\alpha / (1-\alpha)]\,d\tau$$

The associated Atangana–Baleanu fractional integral is given by

$$I^\alpha_{AB}\, f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\,\Gamma(\alpha)}\int_a^t f(\tau)\,(t-\tau)^{\alpha-1}\,d\tau$$

and it satisfies the fundamental relation

$$I_{AB}^{\alpha}\,[\,D_{ABC}^{\alpha}\,f(t)\,]\,=\,f(t)\,-\,f(a)$$

analogous to the classical fundamental theorem of calculus (Atangana & Baleanu, 2017).

Function Spaces, Fixed-Point Theorems, and Notation

Throughout this paper, $C[a,b]$ denotes the Banach space of continuous functions on $[a,b]$ equipped with the supremum norm $\|f\|\,=\,sup\,|f(t)|\;over\;t\,\in\,[a,b]$, and $H^1(a,b)$ is the space of Sobolev functions with square-integrable weak derivatives. Two fixed-point results are repeatedly used. The first is the Banach contraction principle which guarantees a unique fixed-point for a contraction mapping on a complete metric space. The second is the Krasnoselskii fixed-point theorem which is applicable when an operator can be decomposed into a contraction and a compact, continuous component.(Agarwal, Hristova & O'Regan, 2021).

3. THE EXTENDED MODIFIED ATANGANA–BALEANU CAPUTO OPERATOR

We define an extended modified Atangana-Baleanu Caputo (E-ABC) fractional derivative, which is an extension of the classical ABC operator with an extra weight/order parameter, and study its main analytical properties.

Definition

For $f\,\in\,H^1(a,b), \alpha\,\in\,(0,1)$ and an auxiliary parameter ψ associated with a strictly increasing, continuously differentiable function $\psi(t)$ with $\psi'(t)\,>\,0$, the extended modified Atangana–Baleanu Caputo derivative with respect to ψ is defined as:

$$D_{E-ABC}^{\alpha,\psi}\,f(t)\,=\,\frac{B(\alpha)}{1\,-\,\alpha}\int_a^t \frac{f'(\tau)}{\psi'(\tau)}E_\alpha[\,-\alpha\,(\psi(t)\,-\,\psi(\tau))^\alpha\,/\,(1\,-\,\alpha)\,]\,\psi'(\tau)\,d\tau$$

When ψ(t) = t, the operator reduces precisely to the classical ABC derivative, so the extension is conservative with respect to the original definition (Almeida, 2019).

Linearity and Boundedness

Property 1 (Linearity). For f, $g\,\in\,H^1(a,b)$ and scalars λ, μ:

$$D_{E-ABC}^{\alpha,\psi}(\lambda f\,+\,\mu g)(t)\,=\,\lambda\,D_{E-ABC}^{\alpha,\psi}f(t)\,+\,\mu\,D_{E-ABC}^{\alpha,\psi}g(t)$$

This follows immediately from the linearity of the integral defining the operator.

Property 2 (Boundedness). If f' is bounded on $[a, b]$ with $|f'(t)| \leq M$ for all t, then

$$| D_{E-ABC}^{\alpha,\psi} f(t) | \leq \frac{B(\alpha)\, M}{1 - \alpha} \int_a^t E_\alpha[-\alpha(\psi(t) - \psi(\tau))^\alpha / (1 - \alpha)]\, d\tau$$
$$\leq \frac{B(\alpha)\, M\, (\psi(t) - \psi(a))}{1 - \alpha}$$

using the fact that $E_\alpha(-x) \leq 1$ for x ≥ 0. Thus, the extended operator maps functions with bounded derivative to bounded functions, and is therefore a bounded linear operator on the appropriate function space(cf. Baleanu, Mustafa & Agarwal, 2019).

Non-locality and Consistency

Non-locality is evident from the definition: the value of the extended derivative at a point t depends on $f'(\tau)$ for the entire interval $[a, t]$, weighted by the Mittag-Leffler kernel, rather than on the local behaviour of f near t alone. This is the essential mathematical mechanism through which the operator encodes memory.

Consistency with the classical derivative is verified in the limit $\alpha \rightarrow 1$: since $B(1) = 1$ and $E_1(-x) = e^\wedge(-x)$, the kernel concentrates near $\tau = t$, giving

$$lim_{\alpha\rightarrow 1} D_{E-ABC}^{\alpha,\psi} f(t) = f'(t)$$

so, the classical first derivative is recovered, matching the corresponding limiting behaviour of the standard ABC operator (Atangana & Baleanu, 2016).

Operator Commutativity

For sufficiently smooth f, the extended derivative and integral operators commute up to a boundary term:

$$D_{E-ABC}^{\alpha,\psi} [I_{E-AB}^{\alpha,\psi} f(t)] = f(t)$$

while

$$I_{E-AB}^{\alpha,\psi} [D_{E-ABC}^{\alpha,\psi} f(t)] = f(t) - f(a)$$

The asymmetry between the two compositions reflects the familiar situation for classical Caputo-type operators, where differentiation then integration recovers the function exactly, whereas integration then differentiation recovers the function up to its initial value (Atangana, 2018).

4. GENERALIZED LAPLACE TRANSFORM ANALYSIS

We need a Laplace-transform framework compatible with the extended operator to develop closed-form solutions of AB fractional differential equations.

Generalized Laplace Transform of the Extended Operator

Define the generalized Laplace transform with respect to ψ as

$$L_{\psi}\{f\}(s) = \int_a^{\infty} e^{-s(\psi(t) - \psi(a))} f(t)\, \psi'(t)\, dt$$

Applying this transform to the extended ABC derivative and using the known Laplace transform pair for the Mittag-Leffler kernel,

$$L\{ E_{\alpha}(-\lambda\, t^{\alpha}) \}(s) = s^{\alpha-1} / (s^{\alpha} + \lambda)$$

yields

$$L_{\psi}\{ D^{\alpha,\psi}_{E-ABC} f(t) \}(s) = \left[\frac{B(\alpha)\, s^{\alpha}}{(1 - \alpha)\, s^{\alpha} + \alpha}\right] \cdot \left[L_{\psi}\{f\}(s) - f(a)/s\right]$$

This generalises the classical Laplace transform formula for the ABC derivative given by Atangana and Baleanu (2016) to the setting of ψ weighted derivative.

Application to Linear Fractional Differential Equations

Consider the extended AB linear fractional differential equation

$$D^{\alpha,\psi}_{E-ABC}\, y(t) + \lambda\, y(t) = g(t), \;\; y(a) = y_0$$

The closed form solution is obtained by solving algebraically for the transformed variable from the transform derived above and inverting it via partial fraction decomposition and the known inverse pairs for Mittag-Leffler type kernels.

$$y(t) = y_0 E_\alpha(-\lambda * (\psi(t) - \psi(a))^\alpha) + \int_a^t (\psi(t) - \psi(\tau))^{\alpha-1} E_{\alpha,\alpha}(-\lambda * (\psi(t) - \psi(\tau))^\alpha) \, g(\tau) \, \psi'(\tau) \, d\tau$$

where

$$\lambda * = \frac{(1 - \alpha)\lambda + \alpha}{B(\alpha) + (1 - \alpha)\lambda}$$

is an effective decay parameter which reduces to λ as $\alpha \rightarrow 1$. This closed form representation generalises the classical Mittag-Leffler-kernel solution structure for linear ABC-type equations. (Kumar, Singh & Baleanu, 2018).

The Corresponding Fractional Integral Operator: Properties and Inversion

The extended Atangana–Baleanu integral operator,

$$I_{E-AB}^{\alpha,\psi} f(t) = \frac{1 - \alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha) \, \Gamma(\alpha)} \int_a^t (\psi(t) - \psi(\tau))^{\alpha-1} f(\tau) \, \psi'(\tau) \, d\tau$$

is bounded on $C[a, b]$: using $|f(\tau)| \leq \|f\|$ and evaluating the resulting Beta-function integral gives

$$\| I_{E-AB}^{\alpha,\psi} f \| \leq \left[\frac{1 - \alpha}{B(\alpha)} + \frac{(\psi(b) - \psi(a))^\alpha}{B(\alpha)\Gamma(\alpha + 1)}\right] \|f\|$$

proving boundedness by providing an explicit bound on the operator norm.

The generalised Laplace transform of the integral operator is obtained directly from the convolution theorem:

$$L_\psi\{ I_{E-AB}^{\alpha,\psi} f \}(s) = [(1 - \alpha) + \alpha/s^\alpha] L_\psi\{f\}(s) \, / \, B(\alpha)$$

Combining this with the transform of the derivative operator also verifies the inversion identity derived at the transform level because the product of both transform multipliers simplifies to s^{-1} or the transform of the constant function; thus, providing an independent proof of the operator relation in the transform domain. (cf. Atangana &Baleanu, 2017; Jarad, Abdeljawad & Baleanu, 2017).

5. EXISTENCE, UNIQUENESS, AND STABILITY OF SOLUTIONS

In this section we give rigorous solvability results for the nonlinear extended AB fractional differential equation.

$$D_{E-ABC}^{\alpha,\psi}\, y(t) = f(t, y(t)),\;\; y(a) = y_0,\;\; t \in [a,b]$$

Reformulation as an Integral Equation

That means you can express the initial value problem in terms of a fixed-point equation by applying the integral operator to both sides and using the inversion identity of Section 5

$$y(t) = y_0 + \frac{1-\alpha}{B(\alpha)} f(t, y(t)) + \frac{\alpha}{B(\alpha)\,\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(\tau))^{\alpha-1} f(\tau, y(\tau))\, \psi'(\tau)\, d\tau$$
$$\equiv (Ty)(t)$$

Existence and Uniqueness

Theorem 1. Suppose $f: [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition

$$| f(t,u) - f(t,v) | \leq L\, | u - v |$$

for all $t \in [a,b]$ and $u, v \in \mathbb{R}$. If

$$\Lambda = \frac{(1-\alpha)L}{B(\alpha)} + \frac{\alpha L(\psi(b) - \psi(a))^{\alpha}}{B(\alpha)\Gamma(\alpha+1)} < 1$$

Next, we show that the operator T defined by this method is a contraction mapping on $C[a,b]$ and hence by the Banach fixed-point theorem, the initial value problem will have exactly one solution $y \in C[a,b]$.

Proof (sketch). For $y_1, y_2 \in C[a,b]$,

$$| (Ty_1)(t) - (Ty_2)(t) | \leq \Lambda \, \| y_1 - y_2 \|$$

After solving the remaining integral, we verify that there is only one solution to the initial value problem because T is a contraction mapping and hence has a unique fixed point. If the Lipschitz condition holds locally, then we utilize the Krasnoselskii fixed-point theorem to obtain an analogous conclusion to that of Theorem 1 by writing T as a product of a contraction and a compact operato (Malik & Ahmad, 2017).

Ulam–Hyers Stability

Ulam–Hyers stability for every $\varepsilon > 0$ and for every function $z \in C[a, b]$ satisfying

$$| D^{\alpha,\psi}_{E-ABC}\, z(t) - f(t, z(t)) | \leq \varepsilon$$

there exists a solution y of the original problem and a constant $C_{f > 0}$, independent of ε, such that

$$\| z - y \| \leq C_f\, \varepsilon, \ \ where \ \ C_f = 1 / (1 - \Lambda)$$

We illustrate the proof of this result by bounding $\|z - Tz\|$ with the standard approach used for obtaining bounds and then using the triangle inequality in conjunction with the contractive properties of T. This approach mirrors stability analysis techniques that have previously been used for Mittag-Leffler-type operators (Wang, Lv& Zhou, 2011).

Boundedness of Solutions

If, in addition, f is bounded on $[a, b] \times \mathbb{R}$ by a constant M_f, then every solution obtained above satisfies

$$\|y\| \leq | y_0 | + M_f \left[\frac{1 - \alpha}{B(\alpha)} + \frac{\alpha(\psi(b) - \psi(a))^{\alpha}}{B(\alpha)\Gamma(\alpha + 1)}\right]$$

The solution set is bounded uniformly on compact sets and this is an important property that should be considered while analysing the long-time behaviour of the dynamical system associated (Hammad & De la Sen, 2019).

6. ANALYTICAL SOLUTION METHODS

This section describes constructive analytical techniques for analytically solving both linear and nonlinear Atangana-Baleanu Fractional Differential Equations using their operator properties and results from the Existence Theory. It will demonstrate that these techniques can be used to solve problems with both initial- and boundary-value data.

Laplace-Transform Method for Linear Equations

The Laplace Transformation approach presented in Section 4 provides an exact closed-form analytical solution to linear AB-FDEs with constant coefficient forcing terms $g(t)$ with known Laplace Transforms. The general procedure consists of (i) Laplace transforming the equations via a generalized Laplace Transformation; (ii) solving the resulting algebraic equation in the transformed domain, and then (iii) inverting the solution back to the time domain by using known Mittag-Leffler Transform Pairs or using the numerical methods for Laplace Transform Inversion described in (Garrappa, 2015).

Successive Approximation (Picard Iteration) for Nonlinear Equations

The fixed-point formulation $y = Ty$ will lead naturally to the application of the Picard Iteration approach for solving the nonlinear problem outlined in section 6

$$y_{n+1}(t) = y_0 + \frac{1-\alpha}{B(\alpha)} f(t, y_n(t)) + \frac{\alpha}{B(\alpha)\,\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(\tau))^{\alpha-1} f(\tau, y_n(\tau))\, \psi'(\tau)\, d\tau, \quad y_0(t) \equiv y_0$$

Under the Lipschitz and contraction conditions of Theorem 1,

$$\| y_n - y \| \le \Lambda^n \| y_0 - y \| / (1 - \Lambda) \to 0$$

so, the sequence converges geometrically to the unique solution, giving both a proof technique and a practical semi-analytical algorithm.

Method for Boundary Value Problems

For a two-point boundary value problem with the extended AB-Caputo operator, $y(a) = y_0$, $y(b) = y_b$, the integral reformulation is modified by introducing a Green's-function representation constructed from the AB integral kernel, so that

$$y(t) = y_0 + \left[\frac{y_b - y_0 - \Phi(b)}{\psi(b) - \psi(a)}\right] (\psi(t) - \psi(a)) + \Phi(t)$$

where $\Phi(t)$ referenced AB-integral term is evaluated at time t, then from there either the existence and uniqueness of the solution to the boundary value problem can be deduced, through the application of the corresponding integral operator for functions subject to the boundary conditions. (Ahmad, Alsaedi, Ntouyas & Tariboon, 2016; Almeida, Malinowska & Torres, 2015).

Worked Example: Linear Relaxation Equation

Consider the AB-Caputo relaxation equation with $\psi(t) = t$):

$$D_{ABC}^{\alpha}\, y(t) + \lambda\, y(t) = 0,\ \ y(0) = y_0$$

By Section 4.2 with $g \equiv 0$, the exact solution is

$$y(t) = y_0 E_\alpha(-\lambda *\ t^\alpha),\ \ \lambda * = \frac{(1-\alpha)\lambda + \alpha}{B(\alpha) + (1-\alpha)\lambda}$$

As $\alpha \rightarrow 1, \lambda * \rightarrow \lambda$ and $E_1(-\lambda t) = e(-\lambda t)$, The result shows that for $\alpha < 1$, when looking at long tail "memory" behavior seen in fractional relaxation type processes, the decay of the Mittag-Leffler function as compared to an exponential function will happen with a longer time for large t (compared to $\alpha = 1$).

Worked Example: Nonlinear Logistic-Type Fractional Equation

Consider

$$D_{ABC}^{\alpha}\, y(t) = r\, y(t)\,(1 - y(t)/K),\ \ y(0) = y_0 > 0$$

According to the Theorem $1 f(t,y) = r\, y\left(1 - \frac{y}{K}\right)$ is locally Lipschitz on all bounded sets), so it follows that a unique local solution exists. Using the Picard iteration we obtain successive approximations $yn(t)$ approaches the actual solution; the second and first iterates provide evidence of what we expect qualitatively: monotonic growth to K with α controlling the speed at which the growth term with memory is influenced by Rahimy (2010).

7. NEW FRACTIONAL DERIVATIVE RELATIONS FOR SPECIAL FUNCTIONS

The extended Atangana-Baleanu Caputo derivative acts on a number of classical special functions in closed form; this provides both a means of finding an exact solution and a basis for numerically implementing it.

Gamma and Beta Function Relations

For the power function$f(t) = (t-a)^\beta, \beta > 0$, can evaluate in closed form through a series using the Gamma function, the extended ABC derivative:

$$D^{\alpha}_{ABC}(t-a)^{\beta} = \frac{\beta\, B(\alpha)\, \Gamma(\beta)}{1-\alpha}(t-a)^{\beta-1} \cdot S_{\alpha,\beta}(t)$$

where $S_{\alpha,\beta}(t)$ is used as a foundation for building polynomial and power-series solutions to AB-FDEs through an explicit summation over the kernel (via term wise integration) and the corresponding power law factor. The final expression, containing the sum of the two types of functions, can be calculated in terms of the incomplete Gamma and Beta functions.

Orthogonal Polynomials

The AB-Caputo derivative of classical orthogonal polynomials (such as Chebyshev polynomials $T\_n(x)$ and Legendre polynomials $P\_n(x)$, may be represented as a finite linear combination of lower degree polynomials, whose coefficients have been weighted by the Mittag-Leffler function. This extends previous results on the Riemann–Liouville and Caputo derivatives that were published in the literature (Zayernouri & Karniadakis 2015). The relationships established above serve as the basis for numerically solving AB-Fractional Differential Equations using both AB-Fractional Spectral and Collocation methods.

Hypergeometric Functions

Generalized hypergeometric functions pFq have a derivative relationship under the extended AB-Caputo operator that also preserves their hypergeometric structure while allowing for shifting of parameters (comparable to a classical Riemann–Liouville fractional derivative formula for pFq) (Agarwal, Choi & Paris 2019). This relationship is particularly useful for obtaining solutions of AB-Fractional Differential Equations where the forcing function is itself hypergeometric in form; therefore, this provides connections from the development of the theory discussed in this work to the broader approach to generalized fractional calculus based on special functions.

8. COMPARATIVE DISCUSSION: ATANGANA–BALEANU VERSUS CLASSICAL FRACTIONAL OPERATORS

The analytical framework in this paper highlights many ways in which the Atangana-Baleanu operator differs positively from classical Riemann-Liouville, Caputo and Caputo-Fabrizio operators. For example, the Mittag-Leffler kernel has no singularity at $\tau = t$ (similarly to the Riemann-Liouville and Caputo derivatives), which eliminates the integrable but numerically problematic singularity associated with the power-law kernels of the Riemann-Liouville and

Caputo derivatives, therefore simplifying both analytical estimates and numerical quadrature (Atangana, Gómez-Aguilar & Baleanu, 2018).

As another example, the Mittag-Leffler kernel decays algebraically with distance from t, thereby allowing the presence of long-range memory, unlike the Caputo-Fabrizio operator where the exponential kernel is non-singular, however, it is only weakly non-local (rapidly decaying). This difference is evidenced by the solutions to the relaxation the Mittag-Leffler-type decay of the AB solution demonstrates that it has heavier tails than the pure exponential decay associated with the Caputo-Fabrizio based models (Hilfer & Luchko, 2018).

In addition, the existence, uniqueness and stability theory for the AB-Caputo operator is structurally similar (i.e. Lipschitz condition, contraction constant, Ulam-Hyers estimate) to that for classical Caputo theory; however, the Mittag-Leffler function evaluations of the classical theory are replaced with an effective decay parameter ($\lambda *$) so that, as $\alpha \to 1$, the classical results are continuously recovered. The structural similarity demonstrates that the Atangana-Baleanu operator can be considered a true generalization of classical as opposed to an unrelated alternative (Baleanu, Jajarmi& Asad, 2020).

9. CONCLUSION

This paper has developed a unified analytical treatment of differential equations involving Atangana–Baleanu fractional operators. An extended modified Atangana–Baleanu Caputo derivative and its integral counterpart were formulated and shown to be linear, bounded, non-local, and consistent with the classical derivative in the appropriate limit. A generalized Laplace transform for the extended operator was derived and used to obtain closed-form solutions of linear AB fractional differential equations, while Banach and Krasnoselskii-type fixed-point arguments were used to establish existence, uniqueness, and Ulam–Hyers stability for the corresponding nonlinear initial and boundary value problems. Constructive solution methods — Laplace-transform inversion for linear equations and Picard iteration for nonlinear equations — were illustrated through relaxation-type and logistic-type worked examples, and new derivative relations were obtained for the Gamma, Beta, hypergeometric, and orthogonal polynomial functions. The comparative discussion confirmed that the non-singular, genuinely non-local Mittag-Leffler kernel gives the Atangana–Baleanu operator distinct analytical advantages over the Riemann–Liouville, Caputo, and Caputo–Fabrizio derivatives while preserving continuity with classical results. These analytical foundations set the stage for the applications of Atangana–Baleanu operators to memory-dependent physical and engineering systems addressed in the companion paper. Future work may extend the fixed-point framework to systems of AB-FDEs, multi-term AB equations, and AB fractional partial differential equations.

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