I-Function-Based Solutions of Fractional Differential Equations and their Applications

Dr. Vinod Kumar Prajapati*

Assistant Professor of Mathematics, PMCoE, S.G.S. Govt. Autonomous P.G. College, Sidhi, Madhya Pradesh, India

drvkp.sohawal@gmail.com

Abstract: Closed-form solutions of generalized fractional kinetic equations with the power function source term are obtained by using Sumudu transform. The I-function of Rathie is a generalization of the H-function of Fox which permits real exponents of the gamma factors in its Mellin–Barnes kernel; this means that one result encompasses the H-function, Meijer's G-function, the Mittag-Leffler family and certain fractional-power analogues of them. First, another I-function, Sumudu image of t^(μ−1) I[λt^ρ] is obtained. We then numerically solve the one-term and the two-term Riemann-Liouville fractional kinetic equation in each case as a convergent series of one higher order I-functions. Specialization of the parameters retrieves the known solutions of Haubold and Mathai, and of Saxena and Kalla, and provides explicit solutions to be expressed in terms of Fox's H-function, confluent hypergeometric functions and Mittag-Leffler-type series. Applying this, we calculate the solution for an I-function source of Mittag-Leffler type, compare it with an other independent numerical solution of the underlying Volterra integral equation, and investigate how the solution behaves with respect to the change of the parameters of the fractional order ν and the I-function exponent B, which respectively govern the decay of the reacting number density and the finite propagation speed.

Keywords: I-function; H-function; fractional kinetic equation; Sumudu transform; Riemann–Liouville fractional integral; Mittag-Leffler function.

1. INTRODUCTION

Fractional differential and integral equations have become significant mathematical models of physical phenomena with memory effects, anomalous diffusion and power-law relaxation. Fractional equations are a more general way of describing systems with hereditary properties and nonlocal behavior, compared to classical integer-order equations [16, 17]. A major application is the fractional kinetic equation, which is used to describe the changes in the composition of the chemical elements in stellar systems, including stars like the Sun. In fractional kinetic models, the ordinary integral which is a part of classical kinetic equations is substituted with the Riemann–Liouville fractional integral of order ν. The evolution of the number density can, therefore, be described using Mittag-Leffler functions, instead of the exponential functions used in the classical kinetic models [4].

The use of generalized special functions has also been used to extend the analytical solution of fractional differential and integral equations. Special functions represented by Mellin–Barnes integrals, especially Fox's H-function and Rathie's I-function, are powerful mathematical tools for solving a number of fractional equations. Rathie's I-function is an extension of the H-function, where the gamma factors in the Mellin–Barnes integrand of the H-Function are raised to real powers. This generalization includes a wider class of special functions, and provides more flexibility in deriving analytical solutions to fractional kinetic equations.

In the present study, analytical solutions of generalized fractional kinetic equations with the source terms of power function and Rathie's I-function are developed by using the Sumudu transform. The investigation starts with a transformation lemma for the function (t^{\mu-1}I[\lambda t^\rho]) which shows that its Sumudu transform can be written as an I-function of a higher order (Lemma 3.1). A series solution is found for one-term fractional kinetic equations in terms of I-functions (Theorem 4.1) extending this result. The analysis is then extended to the case of two term fractional kinetic equations, whose solutions are given in the form of double-series representations (Theorem 4.2).

In addition, important special cases of the generalized solutions involving Fox's H-function, confluent hypergeometric function and Mittag-Leffler-type series are discussed. The special cases make a link with those previously reported [4, 6] and illustrate the general validity of the proposed mathematical formulation. A numerical application is also included in the investigation where the analytic series solution is compared with an independently obtained numerical solution to check the accuracy of the analytic solution. The influence of the fractional order ν and the I-function exponent B on the behavior of the solutions is illustrated with the help of graphical representations. The main aims of this study are to generalize and expand the analytical approach of fractional kinetic equations and to show the effectiveness of Rathie's I-function and the Sumudu transform in fractional calculus and mathematical physics in these theoretical and numerical investigations.

2. LITERATURE REVIEW

Because of the ability to describe complex physical processes that involve memory and hereditary effects, fractional kinetic equations have become the subject of much attention in mathematical physics. [4] presented the fractional kinetic equation and showed that the solutions could be written in terms of Mittag-Leffler functions. The physical meaning of fractional kinetic equations and their wide applications were also examined [18]. [5, 6, 20]

These studies further including the use of Mittag-Leffler functions and power-type functions as source terms. [9] Considered generalized functions of fractional calculus as sources in fractional kinetic equations, adding to this field.

Generalized special functions have been used extensively in finding analytical solutions to fractional differential and kinetic equations as well. [2, 3] Introduced the H-function, which represents a number of classical special functions in a unified way by the Mellin–Barnes integral. Later developments resulted in other generalizations, called I-functions. In the I-function of Saxena, the finite sums are of expressions formed from H-function-type gamma ratios, while in Rathie's I-function [1] the gamma factors can be raised to real powers. 10] Similar studies for incomplete forms, while [15] studied fractional kinetic equations involving the Aleph-function. [1] showed that his I-function contains a few important special functions and mathematical expressions such as polylogarithms, the exact partition function of the Gaussian model, that cannot be generally represented by the H-function. Other contributions by [21] and [12, 13] presented the applications of Rathie's I-function and its incomplete forms in fractional integral operators and fractional integral equations, respectively. Likewise, [11, 14] established exact results in fractional calculus containing I-functions, which brings out the significance of these functions in the mathematical world.

Analytical solutions of fractional kinetic equations have been found by the use of integral transform techniques. In the past, the Laplace transform was the one mainly used in the previous investigations owing to its ability to solve fractional differential and integral equations. [7] proposed the Sumudu transform which has some useful properties such as dimensional consistency and simplified transformation properties of power functions and has a theoretical relationship with the Laplace transform. In recent years, [23] further extended the Sumudu transform approach to solve multi-term fractional kinetic equations with arbitrary source function. These contributions laid the groundwork for the possibility of Sumudu transform as an alternative analytical approach to solve fractional kinetic models.

Although these advancements have occurred, as revealed in the reviewed literature, the following points can be summarized as a research gap in the current literature on the application of Rathie's I-function and Sumudu transform to fractional kinetic equations. Although a number of investigations have been studied in literature involving generalized special functions, fractional operators and Integral Transform methods, none of the studies reviewed here provides a solution to the fractional kinetic equations by taking Rathie's I-function as a source function by the Sumudu transform. The present study will therefore use the Sumudu transform approach to propose analytical solutions to fractional kinetic equations with Rathie's I-function in order to fill this void and to enlarge the mathematical armory of the fractional kinetic equations and its applications in Mathematical Physics.

3. METHODOLOGY

The method consists of three steps: (1) transform the kinetic equation to an algebraic equation using the Sumudu transform, (2) numerically solve the algebraic equation and perform the inverse Sumudu transform term by term, and (3) compare the result with an independent numerical solution.

3.1 Riemann–Liouville fractional integral

The fractional integral of order ν is [16, 17]:

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x)\, dx, \qquad t > 0,\ \mathrm{Re}(\nu) > 0 \qquad (3.1) $$

3.2 Mittag-Leffler functions

The two-parameter Mittag-Leffler function [19] is given in (3.2), with E_α(z) = E_(α,1)(z) and E_1(z) = e^z. It is a special case of Fox's H-function, as shown in (3.3) [3, 19].

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \mathrm{Re}(\alpha) > 0 \qquad (3.2) $$

$$ E_{\alpha,\beta}(-z) = H_{1,2}^{1,1}\left[z \,\middle|\, \begin{matrix} (0,1) \\ (0,1),\ (1-\beta,\alpha) \end{matrix} \right] \qquad (3.3) $$

3.3 The Sumudu transform

The Sumudu transform is given by (3.4). It is a linear convolution, as it is related to the Laplace transform F(p) by G(u) = F(1/u)/u. The solution must have the following two properties: on powers (3.5) and on the Riemann–Liouville integral (3.6) [8, 22, 23].

$$S[f(t)](u) = G(u) = \int_0^\infty e^{-t} f(ut)\, dt = \frac{1}{u}\int_0^\infty e^{-t/u} f(t)\, dt \qquad (3.4)$$

$$S\left[t^\gamma\right](u) = \Gamma(\gamma+1)\, u^\gamma, \qquad S^{-1}\left[u^\gamma\right](t) = \frac{t^\gamma}{\Gamma(\gamma+1)}, \qquad \mathrm{Re}(\gamma) > -1 \qquad (3.5)$$

$$S\left[{}_0D_t^{-\nu} f(t)\right](u) = u^\nu\, G(u) \qquad (3.6)$$

3.4 Rathie's I-function

Rathie [1] introduced the kernel (3.8) and Mellin–Barnes integral (3.7) for the definition of the I-function. In the following, 0 ≤ m ≤ q and 0 ≤ n ≤ p are integers, α_j, β_j > 0, the exponents A_j, B_j are positive reals and the contour $\mathcal{L}$ is the separation of the poles of the numerator gamma products. Under condition (3.9) the integral converges. When all A_j = B_j = 1 it reduces to Fox's H-function [2, 3].

$$I_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_j,\alpha_j;A_j)_{1,p}\\(b_j,\beta_j;B_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \theta(s)\, z^s\, ds \qquad (3.7)$$

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma^{B_j}(b_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma^{A_j}(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma^{B_j}(1-b_j+\beta_j s)\ \prod_{j=n+1}^{p}\Gamma^{A_j}(a_j-\alpha_j s)} \qquad (3.8)$$

$$\Omega = \sum_{j=1}^{n} A_j\alpha_j - \sum_{j=n+1}^{p} A_j\alpha_j + \sum_{j=1}^{m} B_j\beta_j - \sum_{j=m+1}^{q} B_j\beta_j > 0, \qquad |\arg z| < \frac{\Omega\pi}{2} \qquad (3.9)$$

Two special cases (3.10) will be used later. Both are obtained by summing residues at the simple poles of Γ(−s); the second is equal to (3.3) for B = 1 and its contour version requires 0 < αB < 2.

$$I_{0,1}^{1,0}\left[z\,\middle|\,\begin{matrix}-\\(0,1;1)\end{matrix}\right]=e^{-z},\qquad I_{1,2}^{1,1}\left[z\,\middle|\,\begin{matrix}(0,1;1)\\(0,1;1),\ (1-\beta,\alpha;B)\end{matrix}\right]=\sum_{r=0}^{\infty}\frac{(-z)^r}{[\Gamma(\beta+\alpha r)]^B}\qquad(3.10)$$

3.5 Model equation and solution procedure

The generalized fractional kinetic equation for the number density N(t), with source f(t), rate constant c > 0 and order ν > 0, is:

$$N(t)=N_0\,f(t)-c^{\nu}\,{}_0D_t^{-\nu}N(t)\qquad(3.11)$$

In this paper f(t) = t^(μ−1) I[λt^ρ]. There are 4 steps to the solution:

1. Apply the Sumudu transform to both sides; for the fractional integral use the Sumudu transform in (3.6).
2. Using Lemma 3.1 below, find the Sumudu image of the source with 2.
3. 3. Solve the algebraic equation for G(u) and express the solution as a geometric series.
4. 4a) Invert term by term from (3.5) and identify each term as an I-function by the use of (3.7)).

Lemma 3.1. Let ρ > 0, Ω > 0, |arg λ| < Ωπ/2, and let condition (3.12) hold. If this is the case, the Sumudu image of the source is the I-function of one higher order in (3.13).

$$\mathrm{Re}(\mu)+\rho\min_{1\le j\le m}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)>0\qquad(3.12)$$

$$S\left[t^{\mu-1}I_{p,q}^{m,n}[\lambda t^{\rho}]\right](u)=u^{\mu-1}\,I_{p+1,q}^{m,n+1}\left[\lambda u^{\rho}\,\middle|\,\begin{matrix}(1-\mu,\rho;1),\ (a_j,\alpha_j;A_j)_{1,p}\\(b_j,\beta_j;B_j)_{1,q}\end{matrix}\right]\qquad(3.13)$$

Proof. Substitute (3.7) into (3.4). Condition (3.12) lets the contour be chosen with Re(μ + ρs) > 0, and absolute convergence allows the two integrals to be interchanged. The remainder of (3.14) is then the Euler's gamma integral.

$$S\left[t^{\mu-1}I_{p,q}^{m,n}[\lambda t^{\rho}]\right](u)=\int_0^{\infty}e^{-t}(ut)^{\mu-1}\frac{1}{2\pi i}\int_{\mathcal{L}}\theta(s)\,\lambda^{s}(ut)^{\rho s}\,ds\,dt=u^{\mu-1}\frac{1}{2\pi i}\int_{\mathcal{L}}\theta(s)\,\Gamma(\mu+\rho s)\,(\lambda u^{\rho})^{s}\,ds \qquad (3.14)$$

The new factor Γ(μ + ρs) is a numerator factor of type Γ^(A)(1 − a + αs) with a = 1 − μ, α = ρ and A = 1. Adding it to (3.8) raises n and p by one, which is (3.13).

3.6 Numerical verification method

The analytical solutions are tested independently by directly solving equation (3.11) as a Volterra integral equation. Suppose that the uniform grid is given by t_n = nh. Then the product-trapezoidal rule for the weakly singular kernel (corrector of the fractional Adams method) yields the explicit recursion (3.15)

$$N_n=\frac{N_0 f(t_n)-\dfrac{c^{\nu}h^{\nu}}{\Gamma(\nu+2)}\sum_{j=0}^{n-1}a_{j,n}N_j}{1+\dfrac{c^{\nu}h^{\nu}}{\Gamma(\nu+2)}},\qquad t_n=nh, \qquad (3.15)$$

$$a_{0,n}=(n-1)^{\nu+1}-(n-1-\nu)\,n^{\nu},\qquad a_{j,n}=(n-j+1)^{\nu+1}-2(n-j)^{\nu+1}+(n-j-1)^{\nu+1}$$

The agreement with this scheme is a true test, because it does not call for any special-function series or transform. The sums of the analytical series were evaluated using 40-digit arithmetic, in the python language, using the mpmath library and were stopped when a term became less than 10-25 of the running sum.

4. RESULTS

The closed form solution of the kinetic equation with an I-function source is a series solution in I-functions, and is in agreement with an independently obtained numerical solution within 4 × 10^{-5}.

4.1 Main theorems

Theorem 4.1 (one-term equation). Let c > 0, ν > 0, and let the conditions of Lemma 3.1 hold. Then the solution of equation (4.1) is the series (4.2).

$$N(t) - N_0\, t^{\mu-1} I_{p,q}^{m,n}[\lambda t^{\rho}] = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \qquad (4.1)$$

$$N(t) = N_0\, t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{\nu k}\, I_{p+1,q+1}^{m,n+1}\left[\lambda t^{\rho} \,\middle|\, \begin{matrix} (1-\mu,\rho;1),\ (a_j,\alpha_j;A_j)_{1,p} \\ (b_j,\beta_j;B_j)_{1,q},\ (1-\mu-\nu k,\rho;1) \end{matrix}\right] \qquad (4.2)$$

Proof. Let G(u) = SN(t). Taking the Sumudu transform of (4.1) with (3.6) and Lemma 3.1 gives (4.3).

$$G(u)\,[1+(cu)^{\nu}] = N_0\, u^{\mu-1} \frac{1}{2\pi i} \int_{\mathcal{L}} \theta(s)\, \Gamma(\mu+\rho s)\, \lambda^s u^{\rho s}\, ds \qquad (4.3)$$

For 0 < cu < 1, expanding [1 + (cu)^ν]^(−1) as a geometric series gives (4.4).

$$G(u) = N_0 \sum_{k=0}^{\infty} (-1)^k c^{\nu k} \frac{1}{2\pi i} \int_{\mathcal{L}} \theta(s)\, \Gamma(\mu+\rho s)\, \lambda^s\, u^{\mu-1+\nu k+\rho s}\, ds, \qquad 0 < cu < 1 \qquad (4.4)$$

By (3.5), the inverse transform of u^(μ−1+νk+ρs) is t^(μ−1+νk+ρs) / Γ(μ + νk + ρs). Inverting term by term gives (4.5).

$$N(t) = N_0\, t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{\nu k} \frac{1}{2\pi i} \int_{\mathcal{L}} \theta(s)\, \frac{\Gamma(\mu+\rho s)}{\Gamma(\mu+\nu k+\rho s)}\, (\lambda t^{\rho})^s\, ds \qquad (4.5)$$

The factor Γ(μ + νk + ρs) in the denominator is a factor of index j = q + 1 > m with b = 1 − μ − νk and β = ρ. With (3.7) this is (4.2).

Remark 4.1. The added pair of factors gives +ρ and −ρ to each I-function in (4.2), so that each one has the same Ω as the source. The k-th term contributes 1/Γ(μ + νk + ρs), so the series is convergent for any t > 0, and the restriction cu < 1 is lifted by the analytic continuation.

Theorem 4.2 (two-term equation). Let $c_1, c_2 > 0$, ν, ω > 0, and let the conditions of Lemma 3.1 hold. Then the solution of (4.6) is (4.7).

$$N(t) - N_0\, t^{\mu-1} I_{p,q}^{m,n}[\lambda t^{\rho}] = -c_1^{\nu}\, {}_0D_t^{-\nu} N(t) - c_2^{\omega}\, {}_0D_t^{-\omega} N(t) \qquad (4.6)$$

$$N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} \sum_{k=0}^{r} (-1)^r \binom{r}{k} (c_1 t)^{\nu k} (c_2 t)^{\omega(r-k)} \times I^{m,n+1}_{p+1,q+1}\left[\lambda t^{\rho} \,\middle|\, \begin{matrix} (1-\mu,\rho;1),\ (a_j,\alpha_j;A_j)_{1,p} \\ (b_j,\beta_j;B_j)_{1,q},\ (1-\mu-\nu k-\omega(r-k),\rho;1) \end{matrix} \right] \tag{4.7}$$

Proof. The Sumudu transform gives G(u)[1 + (c₁u)^ν + (c₂u)^ω] equal to N₀ times the image (3.13). The double series (4.8) may be expanded and each term inverted, to obtain (4.7).

$$\frac{1}{1+x+y} = \sum_{r=0}^{\infty} \sum_{k=0}^{r} (-1)^r \binom{r}{k} x^k y^{r-k}, \qquad x = (c_1 u)^{\nu},\ y = (c_2 u)^{\omega} \tag{4.8}$$

Setting $c_2 = 0$ in (4.7) leaves only the terms k = r and recovers (4.2).

4.2 Special cases

Corollary 4.1 (Fox H-function source). With all A_j = B_j = 1, the solution (4.2) becomes (4.9). Setting α_j = β_j = ρ = 1, further yields the Meijer G-function case.

$$N(t) = N_0\, t^{\mu-1} \sum_{k=0}^{\infty} (-1)^k (ct)^{\nu k}\, H^{m,n+1}_{p+1,q+1}\left[\lambda t^{\rho} \,\middle|\, \begin{matrix} (1-\mu,\rho),\ (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q},\ (1-\mu-\nu k,\rho) \end{matrix} \right] \tag{4.9}$$

Corollary 4.2 (exponential source). Using the first case of (3.10) with ρ = 1, the source is N₀ t^(μ−1) e^(−λt). (4.10) becomes each function an I-function, via the sum of residues.

$$N(t) = N_0\, \Gamma(\mu)\, t^{\mu-1} \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{\nu k}}{\Gamma(\mu+\nu k)}\ {}_1F_1(\mu;\ \mu+\nu k;\ -\lambda t) \tag{4.10}$$

Corollary 4.3 (classical results). Letting λ → 0 in (4.10) gives (4.11). With μ = 1, this is the solution of Saxena and Kalla [6] and with ν = 1, it is the exponential decay N₀ e^(−ct).

$$N(t) = N_0\, \Gamma(\mu)\, t^{\mu-1} E_{\nu,\mu}(-(ct)^{\nu}) \quad \xrightarrow{\mu=1} \quad N(t) = N_0\, E_{\nu}(-(ct)^{\nu}) \tag{4.11}$$

Corollary 4.4 (Mittag-Leffler-type I-function source). Using the second case of (3.10), the I-function in (4.2) becomes (4.12). To get (4.13), sum its residues and then sum over k. If B is not equal to 1, the source is not an H-function, so this result is not in the class of H-functions.

$$I_{2,3}^{1,2}\left[\lambda t^{\rho}\,\middle|\,\begin{matrix}(1-\mu,\rho;1),\ (0,1;1)\\ (0,1;1),\ (1-\beta,\alpha;B),\ (1-\mu-\nu k,\rho;1)\end{matrix}\right] \qquad (4.12)$$

$$N(t)=N_0\sum_{r=0}^{\infty}\frac{(-\lambda)^r\,\Gamma(\mu+\rho r)}{[\Gamma(\beta+\alpha r)]^B}\,t^{\mu+\rho r-1}\,E_{\nu,\,\mu+\rho r}(-(ct)^{\nu}) \qquad (4.13)$$

Corollary 4.5 (two-term, power source). Letting $\lambda \to 0$ in (4.7) gives the double series (4.14).

$$N(t)=N_0\,\Gamma(\mu)\,t^{\mu-1}\sum_{r=0}^{\infty}\sum_{k=0}^{r}\frac{(-1)^r\binom{r}{k}(c_1 t)^{\nu k}(c_2 t)^{\omega(r-k)}}{\Gamma(\mu+\nu k+\omega(r-k))} \qquad (4.14)$$

4.3 Numerical verification

The test uses the source of Corollary 4.4 with $N_0 = c = \lambda = \mu = \rho = \alpha = \beta = 1$ and $B = 1.5$. The two-term case uses $c_1 = 1$, $\nu = 0.5$, $c_2 = 0.5$ and $\omega = 0.9$.

Table 1: Graph of series solution versus direct numerical solution (3.15) using h = 0.005.

Equation	ν	t	Series (4.13)/(4.7)	Numerical (3.15)	Absolute difference
One-term	0.50	0.5	0.257815	0.257791	2.4×10^{-5}
One-term	0.50	1.0	0.050317	0.050305	1.2×10^{-5}
One-term	0.50	2.0	−0.088906	−0.088911	5.6×10^{-6}
One-term	0.50	5.0	−0.038354	−0.038356	1.7×10^{-6}
One-term	0.75	0.5	0.258305	0.258297	7.3×10^{-6}
One-term	0.75	1.0	−0.001612	−0.001616	4.3×10^{-6}
One-term	0.75	2.0	−0.137443	−0.137445	1.9×10^{-6}
One-term	0.75	5.0	−0.024962	−0.024963	3.3×10^{-7}
Two-term	0.50, 0.90	0.5	0.184162	0.184129	3.3×10^{-5}
Two-term	0.50, 0.90	1.0	−0.009117	−0.009133	1.6×10^{-5}
Two-term	0.50, 0.90	2.0	−0.103851	−0.103858	6.3×10^{-6}

Equation	ν	t	Series (4.13)/(4.7)	Numerical (3.15)	Absolute difference
Two-term	0.50, 0.90	3.0	−0.085810	−0.085813	3.4×10^{-6}

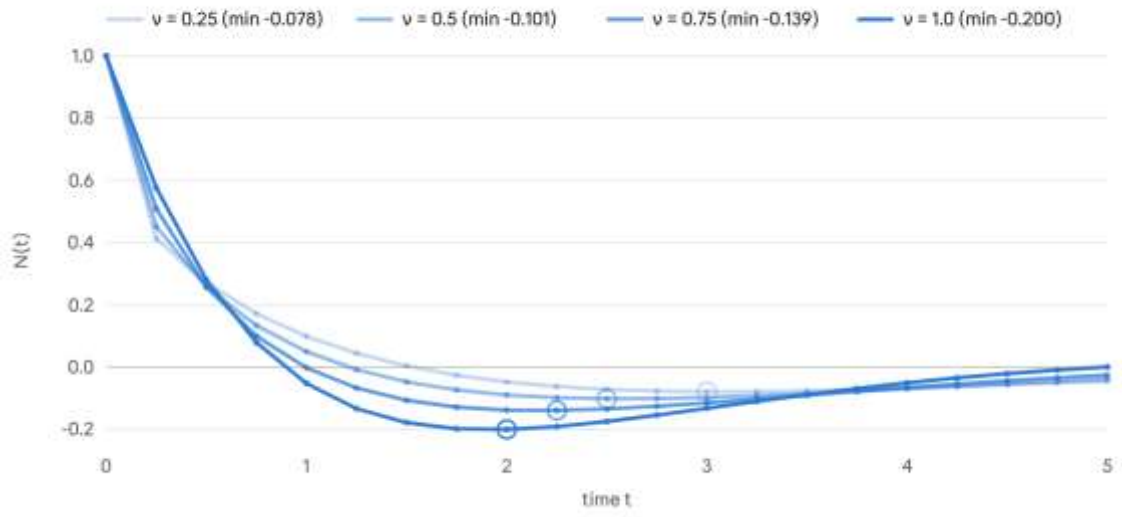
Halving h from 0.02 to 0.0025 at ν = 0.5, t = 1 reduced the difference from 1.03×10^{-4} to 4.2×10^{-6}, about 2.9 times per halving. This is the same order of accuracy as the numerical scheme: O(h^1.5), the missing order of accuracy is not the series' but the error committed by the numerical scheme.

4.4 Effect of the fractional order ν

The smaller the value of ν, the more quickly the solution will drop at first, but then come back to the equilibrium (Figure 1). At t = 0.25 the solution is 0.41 for ν = 0.25 but 0.58 for ν = 1. By t = 5 the ordering reverses: the ν = 1 solution is back near zero (0.001) while the ν = 0.25 solution is still at −0.045. This is the Mittag-Leffler kernel memory effect.

Figure 1. Smaller ν falls faster early but relaxes more slowly

Solution (4.13) for B = 1.5, c = λ = 1; rings mark each curve's minimum

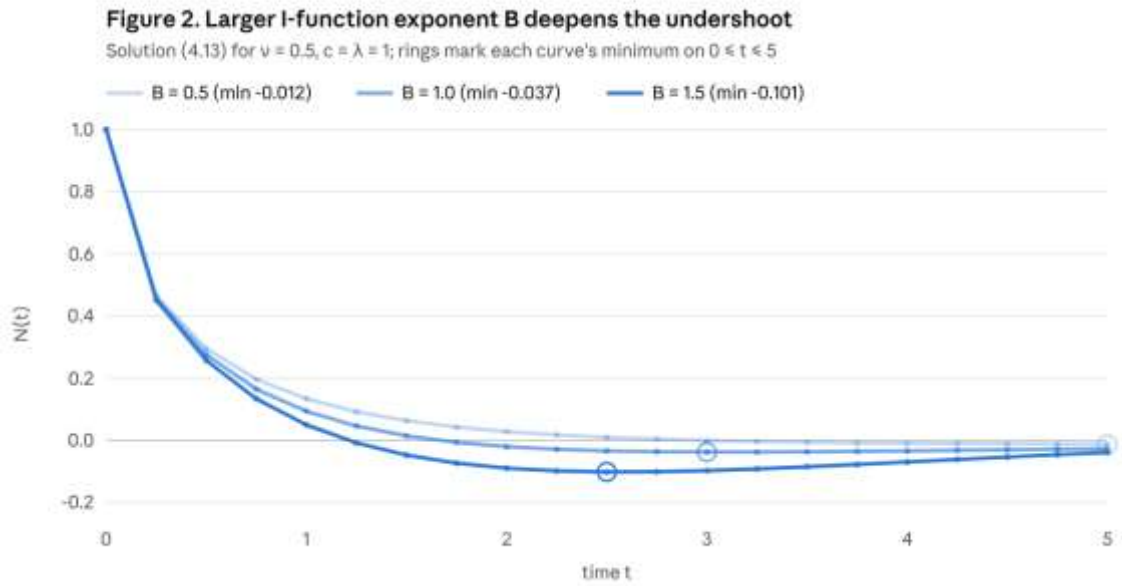


Series (4.13) computed in 40-digit arithmetic, $0 \le t \le 5$, step 0.25

4.5 Effect of the I-function exponent B

The depth of the undershoot (Figure 2) is controlled by the exponent B, which differentiates Rathie's I-function from Fox's H-function. At ν = 0.5 the solution for B = 0.5 falls only to −0.012 by t = 5, while the minimum reaches −0.037 for B = 1 and −0.101 for B = 1.5. This

means that a modeller may have further opportunity to fine-tune the shape of the source by varying B, while maintaining the same order of the kinetics.



Series (4.13) computed in 40-digit arithmetic, $0 \le t \le 5$, step 0.25

The solutions change sign, as does the classical case $\nu = B = 1$ with $c = \lambda = 1$, where (4.13) reduces to $N(t) = N_0(1 - t)e^{(-t)}$. Hence, the value of N(t) is not necessarily a density and should be interpreted as the response of the linear model to the given source.

5. CONCLUSION

The Sumudu transform is used to obtain compact and verifiable solutions for fractional kinetic equations with Rathie's I-function. The Sumudu image of the source is an I-function of higher order than one (Lemma 3.1) and the one-term and two-term equations are solved as a series of I-functions which converge for all $t > 0$, (Theorems 4.1 and 4.2). Due to the possibility to have real exponents on the gamma factors in the I-function, these results contain the Fox H-function, the Meijer G-function, the exponential and the Mittag-Leffler cases, and they give the classical solutions of Haubold and Mathai and Saxena and Kalla. They also provide new solutions, e.g. (4.13), which are not in the H-function class. The formulas were checked in the numerical study by comparing them with a different solver and it was found that the exponent B controlled the depth of the undershoot, while the value of ν controlled the memory of the process. Further research can be done using Hilfer or Atangana–Baleanu operators instead of

the Riemann–Liouville integral, consider the incomplete I-function, and optimize the values of ν and B from measured relaxation data.

References

1. Rathie, A. K. (1997). A new generalization of generalized hypergeometric functions. *Le Matematiche, 52*(2), 297–310.

2. Fox, C. (1961). The G and H functions as symmetrical Fourier kernels. *Transactions of the American Mathematical Society, 98*, 395–429.

3. Mathai, A. M., Saxena, R. K., & Haubold, H. J. (2010). *The H-function: Theory and applications*. Springer.

4. Haubold, H. J., & Mathai, A. M. (2000). The fractional kinetic equation and thermonuclear functions. *Astrophysics and Space Science, 273*, 53–63.

5. Saxena, R. K., Mathai, A. M., & Haubold, H. J. (2002). On fractional kinetic equations. *Astrophysics and Space Science, 282*, 281–287.

6. Saxena, R. K., & Kalla, S. L. (2008). On the solutions of certain fractional kinetic equations. *Applied Mathematics and Computation, 199*, 504–511.

7. Watugala, G. K. (1993). Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *International Journal of Mathematical Education in Science and Technology, 24*(1), 35–43.

8. Belgacem, F. B. M., Karaballi, A. A., & Kalla, S. L. (2003). Analytical investigations of the Sumudu transform and applications to integral production equations. *Mathematical Problems in Engineering, 2003*(3), 103–118.

9. Chaurasia, V. B. L., & Pandey, S. C. (2008). On the new computable solution of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions. *Astrophysics and Space Science, 317*, 213–219.

10. Bansal, M. K., Kumar, D., Harjule, P., & Singh, J. (2020). Fractional kinetic equations associated with incomplete I-functions. *Fractal and Fractional, 4*(2), Article 19. https://doi.org/10.3390/fractalfract4020019

11. Mishra, J. (2018). A remark on fractional differential equation involving I-function. *The European Physical Journal Plus, 133*, Article 36. https://doi.org/10.1140/epjp/i2018-11897-y

12. Bhatter, S., Nishant, Shyamsunder, Purohit, S. D., & Suthar, D. L. (2023). A study of incomplete I-functions relating to certain fractional integral operators. *Applied Mathematics in Science and Engineering, 31*(1), Article 2252996. https://doi.org/10.1080/27690911.2023.2252996

13. Bhatter, S., Jangid, K., Kumawat, S., & Baleanu, D. (2023). Analysis of the family of integral equation involving incomplete types of I and Ī-functions. *Applied Mathematics in Science and Engineering, 31*(1), Article 2165280. https://doi.org/10.1080/27690911.2023.2165280

14. Singh, Y. (2020). A study on fractional differintegrations in association with I-function. *Journal of Mathematics and Computer Science, 10*, 68–77. https://scik.org/index.php/jmcs/article/view/4295

15. Choi, J., & Kumar, D. (2015). Solutions of generalized fractional kinetic equations involving Aleph functions. *Mathematical Communications, 20*(1), 113–123. https://hrcak.srce.hr/140395

16. Podlubny, I. (1999). *Fractional differential equations*. Academic Press.

17. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations* (North-Holland Mathematics Studies, Vol. 204). Elsevier.

18. Saichev, A. I., & Zaslavsky, G. M. (1997). Fractional kinetic equations: Solutions and applications. *Chaos, 7*, 753–764.

19. Gorenflo, R., Kilbas, A. A., Mainardi, F., & Rogosin, S. (2020). *Mittag-Leffler functions, related topics and applications* (2nd ed.). Springer.

20. Saxena, R. K., Mathai, A. M., & Haubold, H. J. (2004). On generalized fractional kinetic equations. *Physica A: Statistical Mechanics and Its Applications, 344*, 657–664.

21. Jangid, K., Bhatter, S., Meena, S., Baleanu, D., Al Qurashi, M., & Purohit, S. D. (2020). Some fractional calculus findings associated with the incomplete I-functions. *Advances in Difference Equations, 2020*. https://doi.org/10.1186/s13662-020-02725-7

22. Katatbeh, Q. D., & Belgacem, F. B. M. (2011). Applications of the Sumudu transform to fractional differential equations. *Nonlinear Studies, 18*(1), 99–112.

23. Chaurasia, V. B. L., & Singh, J. (2011). Application of Sumudu transform in fractional kinetic equations. *General Mathematics Notes, 2*(1), 86–95. https://emis.univie.ac.at/journals/GMN/yahoo_site_admin/assets/docs/Chaurasia_Paper-9.6684008.pdf