

Formation and examination of a few new iterative methods for nonlinear equation numerical solutions

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Abstract

The extensive use of numerical solutions to nonlinear equations in fields as diverse as physics, engineering, economics, and more makes this a vital topic in scientific computing and applied mathematics. Although efficient, classical methods such as Newton-Raphson and Secant methods have drawbacks include being sensitive to initial estimations, diverging in some circumstances, and requiring more computing power to evaluate derivatives. Our goal in this research was to find better iterative solutions to nonlinear equations of the type $f(x)=0$ that are more resilient and have a faster convergence rate. We came up with a few new approaches and tested them out. By adding weight functions and higher-order correction terms to conventional schemes, the newly created approaches are able to adapt and expand upon them. Proofs of convergence order, error estimates, and stability criteria are all part of the extensive theoretical analysis that is given. These techniques improve computing efficiency by achieving higher-order convergence (up to fourth and sixth order) without increasing the number of function or derivative evaluations each iteration. The performance of the suggested approaches is compared to current techniques through a series of numerical tests performed on benchmark nonlinear equations. These novel iterative approaches show good promise as instruments for addressing complicated nonlinear problems in real-world applications, thanks to their enhanced convergence behaviour.

Keywords: Formation, Iterative Methods, Nonlinear Equation, Numerical Solutions

1. INTRODUCTION

When describing occurrences on a grand scale in terms of physical phenomena, such as laws of nature, mathematical models showing relationships between variables are frequently employed. It is essential to obtain the solution of a model in order to investigate the associated physical process and predict its future evolution. It is typically not feasible to obtain the exact or analytical form of answers due to the inherent non-linearity among the variables in the developed models. The study of nonlinear issues presents a greater challenge compared to linear ones. It should be noted that algebraic solutions are usually not discovered for polynomial equations of arbitrary type with coefficients of degree five or above, as stated in the Abel-Ruffini theorem. But remember that the precise responses aren't always what you

desire. Just by numerically approximating the answers to a certain precision, we can gain a deeper grasp of the underlying physical process (Acton, F. S. 2020).

Numerical analysis is a subfield of mathematics concerned with determining optimal numerical solutions to a wide range of issues. The usual method for solving nonlinear equations is to use the concept of iterative theory. In mathematics, iterative techniques are procedures that, given a sufficiently accurate initial estimate, provide successive estimates that, taken together, may converge to a solution. Convergence of generated sequences and the construction of iterative algorithms are the principal foci of mathematical analysis. It is critical to develop fast and efficient iterative algorithms since the complexity of modern mathematical problems is increasing exponentially. Univariate nonlinear equations or multivariate systems of nonlinear equations might provide a mathematical representation of these problems. Using numerical approaches, we can solve the following nonlinear problems (Canale, R. P. 2006):

❖ **Isentropic supersonic flow problem**

Waves compressing and expanding in flow channels are common occurrences in fluid dynamics. A differential equation governs the examination of the uniform two-dimensional isentropic supersonic fluid flow around a sharp convex corner of a frictionless surface:

$$d\delta = -\frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma - 1}{2}M^2} \frac{dM}{M},$$

In where δ is the specific heat ratio of the fluid, M is the Mach number, and δ is the flow deflection angle. From an initial Mach number M_1 to a final Mach number M_2 , the flow deflection angle δ that expands the fluid may be obtained by integrating the above equation (Li, W. 2006).

$$\delta = b^{1/2} \left[\tan^{-1} \left(\frac{M_2^2 - 1}{b} \right)^{1/2} - \tan^{-1} \left(\frac{M_1^2 - 1}{b} \right)^{1/2} \right] - \tan^{-1} (M_2^2 - 1)^{1/2} + \tan^{-1} (M_1^2 - 1)^{1/2},$$

Where $b = \frac{\gamma + 1}{\gamma - 1}$. Given the values of M_1 , γ , and δ , the estimate of M_2 is a specific nonlinear issue in this scenario. Nevertheless, it appears that solving this problem algebraically is not possible in this case.

❖ Three-body problem under gravitational field

For a tiny mass object (m) subjected to the gravitational pull of two large bodies (M_1 and M_2) in orbit around it, the Lagrange points in space represent the sites of equilibrium in the field of celestial mechanics. Lagrange points are locations where the centripetal force needed to propel a tiny object around its orbit is exactly equivalent to the gravitational forces exerted by two big masses. As an example, a satellite can maintain a synchronous orbit with Earth by harnessing the centripetal force provided by the Sun and Earth's gravitational pull. Equation that follows from this model is (Basu, D. 2008).:

$$\frac{GmM_1}{(R \pm r)^2} \pm \frac{GmM_2}{r^2} = m\omega^2(R \pm r),$$

The situation where G stands for the gravitational constant, R for the distance between Earth and the Sun, r for the distance between the satellite and Earth, ω for the angular velocity, and M_1 , M_2 , and m for the masses of the Sun, Earth, and the satellite, respectively. Then, using the Kepler's third law: $\omega^2 = \frac{G(M_1+M_2)}{R^3}$ the given problem leads to solve the following quintic polynomial equation:

$$c_1r^5 + c_2r^4 + c_3r^3 + c_4r^2 + c_5r + c_6 = 0, \quad (1)$$

Where $c_1 = \pm(M_1 + M_2)$, $c_2 = \pm 3(M_1 + M_2)R$, $c_3 = \pm 3(M_1 + M_2)R^2$, $c_4 = (M_2 \mp M_1)R^3$, $c_5 = 2M_2R^4$, $c_6 = M_2R^5$. Two distinct solutions to the provided problem are represented by the +/- symbol in this case. The Lagrange points can be determined by calculating the quintic equation (1) given the specific values of the masses M_1 , M_2 , m , R , and r , as well as the angular velocity ω .

2. SOME BASIC CONCEPTS

The basic concepts that are relevant to this study are presented here.

2.1 Root or zero

A given function is denoted as $f: R \rightarrow R$. A root of an equation is a number that, when multiplied by itself, gives the equation $f(x) = 0$. To rephrase, the zero of the function $f(x)$ is $x = x$. When the x -axis intersects the graph of $y = f(x)$ on the xy -plane, we say that this point is

a geometric zero of $f(x)$. Rather than limiting oneself to the domain of real numbers, the concept of root or zero may be easily extended to functions defined over the complex domain C (Barati, A. 2007).

2.2 Multiple zeros

A solution x^* of $f(x) = 0$ is a zero of multiplicity m of $f(x)$ if, for $x \neq x^*$, $f(x) = (x - x^*)^m g(x)$ where $\lim_{x \rightarrow x^*} g(x) \neq 0$. In particular, if the given function is m times continuously differentiable in the interval $[a, b]$, i.e. $f \in C^m[a, b]$, then it has a zero of multiplicity m at $x^* \in (a, b)$ if and only if

$$f(x^*) = f'(x^*) = f''(x^*) = \dots = f^{(m-1)}(x^*) = 0; \text{ but } f^{(m)}(x^*) \neq 0.$$

If $m = 1$, the given zero is particularly called the simple zero.

2.3 Intermediate value property

Assume that is a continuous function on the interval $[a, b]$ for every x . Refer to Figure 1 for reference, and find an integer s such that $f(a) < s < f(b)$. Then, for any x in the range $[a, b]$, $f(x)$ equals s . One important consequence is that it confirms the presence of a zero of $f(x)$ in the interval $[a, b]$ if the condition is true for $s = 0$ (Hernández, M. A. 2000).

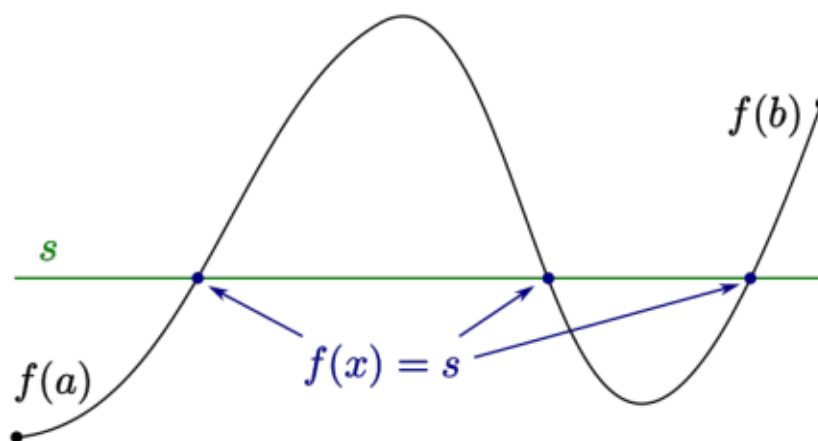


Figure 1: Intermediate value property

2.4 Fixed point iteration

The function's fixed point is a numerical value denoted as x^* . $f(x)$, if $f(x^*) = x^*$. The application of a mapping f to x^* results in x^* being mapped onto it. It should be borne in mind realize that locating a function's zeros is analogous to locating the fixed points of a related function. Specifically, let's say that ϕ is the function that provides the solutions to the equation $f(x) = 0$, where $\phi(x) = x - f(x)$. From that point on, the zero x of $f(x)$ is determined to be the fixed point of $\chi(x)$. Consider any point x_0 . Afterwards, the technique for fixed-point iteration is stated as follows:

$$x_{k+1} = \phi(x_k), \quad k = 0, 1, 2, \dots,$$

where ϕ is called iteration function. This finding proves that there are enough variables for an iterative sequence to converge. $\{x_0, x_1, x_2, \dots\}$ or $\{x_k\}_{k \geq 0}$.

In accordance with Theorem 1.2.1, let x be the interval I value of the zero of the function $f(x)$. In the set I , there are continuous functions $\phi(x)$ and $\phi'(x)$ where the definition of $\phi(x)$ is such that $x = \phi(x)$ is equivalent to $f(x) = 0$. In the case when every integer x in I has an absolute value less than 1, the sequence of approximations $\{x_k\}_{k \geq 0}$ will eventually converge to the root x , provided that x_0 is an integer (Schnabel, R. B. 2013).

2.5 Order of convergence

An iterative sequence $\{x_k\}_{k \geq 0}$ "converges" to an element x^* of rank $p \in \mathbb{R}^+$ when

$$\lim_{k \rightarrow \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = c,$$

for all values of $c > 0$. The asymptotic error constant is the name given to the constant c in this context. Linear convergence, quadratic convergence, and cubic convergence are terms used to describe the aforementioned sequence for $p=1, 2$, and 3 , respectively. Furthermore, let

$e_k = x_k - x^*$ be the errors in corresponding iterations, then the following relation:

$$e_{k+1} = ce_k^p,$$

known as the error equation. Keep in mind that finding the error equation is the first step in figuring out the order of convergence, p . E. Schroder offers an alternative definition of order of convergence. An iteration function $\phi(x)$, corresponding to the iterative scheme

$x_{k+1} = \phi(x_k)$, is said to possess convergence order $p \in \mathbb{Z}^+$, if the sequence $\{x_k\}_{k \geq 0}$ converges to x^* and the following holds:

$$\phi(x^*) = 0; \phi^{(i)}(x^*) = 0 \text{ for } i = 1, 2, \dots, p-1; \phi^{(p)}(x^*) \neq 0.$$

Here, $\phi^{(p)}(x^*)/x^{*!}$ will be the asymptotic error constant.

3. DETERMINATION OF INITIAL APPROXIMATION

The convergence of iterative techniques is strongly affected by the initial approximation of the solution, so determining it is of the utmost importance. Below are some of the methods that may be used to choose good first estimations (Ben-Israel, A. 2017).

3.1 Geometrical approach

By drawing a graph of the function $f(x)$ and finding its intersection with the x -axis, we may obtain an approximation of the solution to the equation $f(x) = 0$. Here we get a very approximate root, as this is the value at which $f(x) = 0$. The lack of precision makes graphical methods for computing the roots almost useless. On the other hand, numerical approaches can use the preliminary estimates from graphical methods as their starting points. The bisection method and other bracketing techniques can be used to an interval that contains the answer.

3.2 Trial and error approach

Here, we take a wild guess at x and see if our estimated root can be better estimated by seeing whether $f(x)$ is near zero. If not, we make another wild guess and see if our updated root estimate holds up. The method is iterated until an estimate is made that yields a $f(x)$ value around zero.

3.3 Incremental search approach

Beginning at one end of the region of interest, this approach evaluates functions at tiny intervals across the region. It is presumed that a root is within the increment whenever the function's sign changes. As a starting point for the bracketing methods, the values of x at the start and end of increment would be useful.

3.4 Sigmoid function approach

Took a time-saving technique to choose a first guess that was close enough to the target solution. Let us pretend that there is a single solution to the function $f(x)$ on the interval $[a,b]$ and that it is continuous on that interval. The function $f^{[\beta]}(x) = \tanh(\beta f(x))$ is called the sigmoid transform of the function $f(x)$ for some $\beta > 0$. Then, for the sufficiently large values of β , the following formula provides approximation to the solution of $f(x) = 0$ as (Momani, S. 2015):

$$x_0 = \frac{1}{2} \left[a + b + \operatorname{sgn}(f(a)) \int_a^b f^{[\beta]}(x) dx \right], \quad (3.1)$$

Where $f(a)f(b) < 0$ and $\operatorname{sgn}(t)$ is the signum function which is defined by

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

In equation (3.1), any rule for numerical integration can be used to evaluate the integral.

4. IM FOR MULTIPLE ROOTS

A solution x^* of $f(x) = 0$ is a root of multiplicity m if and only if

$$f(x^*) = f'(x^*) = f''(x^*) = \dots = f^{(m-1)}(x^*) = 0 \text{ and } f^{(m)}(x^*) \neq 0.$$

Then $f(x)$ can be expressed as

$$f(x) = (x - x^*)^m g(x), \quad (4.1)$$

Where $g(x)$ is a continuous function such that $g(x^*) \neq 0$.

The numerical methods for locating multiple zeros of functions can be broadly classified in two categories in accordance with:

1. Multiplicity 'm' is known,
2. Multiplicity 'm' is not known.

Case (i): When m is known, the well-known Newton method (1.3.2) generates a sequence of iterates $\{x_k\}_{k \geq 0}$, for suitably chosen initial approximation x_0 , wherein the error equation is given by

$$e_{k+1} = \frac{m-1}{m} e_k + O(e_k^2), \quad (4.2)$$

Where $e_k = x_k - x^*$ is the error at k th iteration. For $m > 1$, the sequence $\{x_k\}$ converges to x^* with the rate of a geometric progression with common ratio $\frac{m-1}{m}$. This slows down the convergence order of Newton method from quadratic to linear. To retain the quadratic convergence of Newton method, proposed a modified Newton method given by.

$$x_{k+1} = x_k - m \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots \quad (4.3)$$

The method (1.8.3) converges quadratically for all $m \in \mathbb{N}$.

Case (ii): When m is not known, demonstrated that Newton method as well as secant method converge linearly for the case of multiple roots. They suggested to define a new function,

$$u(x) = \frac{f(x)}{f'(x)}.$$

Since, $f(x) = (x - x^*)^m g(x)$, therefore $u(x)$ can be expressed as

$$u(x) = \frac{(x - x^*)^m g(x)}{m(x - x^*)^{m-1} g(x) + (x - x^*)^m g'(x)} = (x - x^*)v(x), \quad (4.4)$$

Where $v(x) = \frac{g(x)}{m g(x) + (x - x^*)g'(x)}$. Since, $g(x^*) \neq 0$, it directly follows that

$$v(x^*) = \frac{1}{m} \neq 0.$$

Hence, x^* is a straightforward zero of u . All the concepts pertaining to simple zeroes may be used if u and its derivatives are substituted for f and its derivatives in any IM. Take the Newton technique as an example; when we swap out f for u , we get

$$x_{k+1} = x_k - \frac{u(x_k)}{u'(x_k)} = x_k - \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)}. \quad (4.5)$$

With respect to the number of zeros in f , the scheme given by (1.8.5) will always be quadratically convergent. An extra assessment of $f''(x)$ results in a more arduous technique for computing the iterates, which is the single theoretical shortcoming of this method. Using the weight function approach, researchers have recently constructed several higher Order IM for many roots. In the most basic scenario, think of a one-point Newton-like approach with an extra weight function ϕ functioning as (Wheatley, P. O. 2008).

$$x_{k+1} = x_k - \phi(m) \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots, \quad (4.6)$$

Where the function ϕ is directly dependent on the variable m . The highest possible order may be achieved by evaluating the criteria for the function ϕ in the context of procedure (1.8.6). Using the right weight functions in the next stages to create the higher-order methods, the given IM may be easily expanded to a multipoint methodology.

5. FRACTIONAL IM

Using operators with derivatives of any order, fractional calculus expands the idea of elementary calculus to a broader theoretical framework. More accurately describing the spatial non-locality and temporal history of complex physical events are models based on fractional derivatives than standard mathematics. A new field of study is the use of fractional calculus to iteratively solve nonlinear equations. The iterative methods that are created by using the idea of fractional order derivatives are collectively referred to as fractional IM.

5.1 Caputo and Riemann-Liouville fractional derivatives

Several definitions of the fractional order derivative may be found in published works. Among the most common categories of explanations are:

1. Caputo fractional derivative,
2. Riemann-Liouville fractional derivative.

For given $a \in \mathbb{R}$ and the function $f : [a, \infty) \rightarrow \mathbb{R}$, the fractional derivative of f of order $\alpha \in \mathbb{R}$ which is denoted as F_α and is defined as follows:

$$\text{Caputo derivative : } {}_C\mathcal{F}_\alpha(f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \frac{d^n}{dx^n} f(t) dt.$$

$$\text{Riemann-Liouville derivative : } {}_{RL}\mathcal{F}_\alpha(f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt.$$

where n is an integer such that $n-1 < \alpha \leq n$. Apart from these, some other definitions are also available in literature. However, defined in any manner, all of these might not lead to the same results even for the sufficiently smooth functions. Indeed, for a constant function $f(x) = c$, it turns out that ${}_C\mathcal{F}_\alpha(f)(x) = 0$, whereas ${}_{RL}\mathcal{F}_\alpha(f)(x) \neq 0$.

Recent years have seen the development of iterative approaches that use the Caputo or Riemann-Liouville derivatives in place of the conventional derivatives. Nevertheless, to fully resolve the provided integrals, it is necessary to evaluate special functions such as Gamma and Mittag-Leffler functions. Therefore, solving nonlinear equations efficiently using these approaches is not only wasteful, but may be impossible in some instances.

5.2 Conformable fractional derivative

Put forward a well-behaved and straightforward definition, the conformable fractional derivative. A function $f: [a, \infty) \rightarrow \mathbb{R}$ is defined as the conformable derivative of order α for all $a \in \mathbb{R}$ and $\alpha \in (0, 1)$.

$$\mathcal{F}_\alpha^a(f)(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon(x-a)^{1-\alpha}) - f(x)}{\varepsilon}. \quad (5.1)$$

The function f is referred to as α -differentiable if the provided limit exists. Take note that, specifically for $\alpha = 1$, Eq. (6.1) simplifies to the definition of ordinary derivative. Notably, unlike the Caputo and Riemann-Liouville derivatives, conformable derivatives inherit most of the fundamental features of ordinary derivatives, such as the product rule, quotient rule, chain rule, mean value theorems, etc (Gerlach, J. 2014).

Recent work has suggested an iterative strategy for determining the zeros of function $f(x)$ using the notion of conformable fractional derivative (1.9.1). Specifically, the scheme may be expressed as follows for each $\alpha \in (0, 1]$ and $a \in \mathbb{R}$:

$$x_{k+1} = a + \left((x_k - a)^\alpha - \alpha \frac{f(x_k)}{\mathcal{F}_\alpha^a(f)(x_k)} \right)^{1/\alpha}, \quad k = 0, 1, 2, \dots, \quad (5.2)$$

On which f may be expressed as a function that is α -differentiable. Keep in mind that, when $\alpha = 1$, the Newton technique is reduced to the formulation (6.2). Here we lay out the circumstances in which the proposed method yields the quadratic convergence order (Sharma, 2007).

6. SYSTEMS OF NONLINEAR EQUATIONS

Here we have a generic system of n equations with n unknowns, written in the form,

$$\begin{aligned}
 f_1(x_1, x_2, \dots, x_n) &= 0, \\
 f_2(x_1, x_2, \dots, x_n) &= 0, \\
 &\vdots \\
 f_n(x_1, x_2, \dots, x_n) &= 0,
 \end{aligned}
 \tag{6.1}$$

as f_i maps a vector $(x_1, x_2, \dots, x_n)^T$ from the n -dimensional space \mathbb{R}^n onto the real line \mathbb{R} , where $1 \leq i < n$. If there is a nonlinear function in the system, we say that it is nonlinear. By using the compact notation $(f_1, f_2, \dots, f_n)^T$, we can define a vector function F that maps \mathbb{R}^n into \mathbb{R}^n and use it to describe the system of n equations with n unknowns.

$$F(x_1, x_2, \dots, x_n) = (f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n))^T.$$

Thus, system (1.10.1) reduces to the form,

$$F(x) = 0,
 \tag{6.2}$$

Where $x = (x_1, x_2, \dots, x_n)^T$ and the functions $f_1(x), f_2(x), \dots, f_n(x)$ represent the coordinate functions of $F(x)$. The vector $x^* = (x^*_1, x^*_2, \dots, x^*_n)^T$ which satisfies,

$$f_1(x^*) = 0, f_2(x^*) = 0, \dots, f_n(x^*) = 0,$$

Is known as the solution or root of the equation $F(x) = 0$. Keep in mind that the domain \mathbb{R}^n is just used for simplicity's sake, but the concept may be generalized to the case of complex domain \mathbb{C}^n in a similar way.

6.1 Differentiation of vector-valued functions

For all natural numbers from 1 to n , let $L(\mathbb{R}^n)$ be the space of bounded linear mappings. The following matrix form is given for the first Frechet's 1 derivative of $F(x)$, which is a linear operator $F'(x) \in L(\mathbb{R}^n)$, for every $x \in \mathbb{R}^n$.

$$F'(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}, \quad (6.3)$$

Where $\frac{\partial f_i}{\partial x_j}(x)$, for each $i, j \in \{1, 2, \dots, n\}$, is the partial derivative of f_i at x , evaluated with regard to x_j . The Jacobian matrix is the usual name for the integral of $F_0(x)$ in Equation (7.3).

Typically, the k -linear function, described below, is the k -th derivative of F at $x \in \mathbb{R}^n$.

Explanation : An open convex set Ω is defined by $F : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, where F is a sufficiently differentiable function. A k -linear function $F^{(k)}(x)$ is the k th Frechet derivative at $x \in \Omega$ $\mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that the following holds for any vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$:

1. $F^{(k)}(x)(v_1, \dots, v_k) \in \mathbb{R}^n$.
2. $F^{(k)}(x)(v_1, \dots, v_{k-1}, \cdot) \in \mathcal{L}(\mathbb{R}^n)$.
3. $F^{(k)}(x)(v_1, \dots, v_k) = F^{(k)}(x)(v_{\sigma_1}, \dots, v_{\sigma_k})$ for any permutation $\sigma \in \{1, \dots, k\}$.

In view of the above definition, the following notation can be used for convenience:

1. $F^{(k)}(x)(v_1, v_2, \dots, v_k) = F^{(k)}(x)v_1v_2 \cdots v_k$.
2. $F^{(k)}(x)v^{k-1}F^{(p)}(x)v^p = F^{(k)}(x)F^{(p)}(x)v^{p+k-1}$ for any $v \in \mathbb{R}^n$ and $p \in \mathbb{N}$.

6.2 Generalized Taylor series

The proposed theorem defines the generalized Taylor series expansion of a vector-valued function that is sufficiently differentiable:

Assumption: The mapping $F: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is k -times, let's pretend. If the function is Frechet differentiable in a closed convex set $\Omega \in \mathbb{R}^n$, then the given outcome is true for every $x, t \in \Omega$.

$$F(x+t) = F(x) + F'(x)t + \frac{1}{2!}F''(x)t^2 + \dots + \frac{1}{(k-1)!}F^{(k-1)}(x)t^{k-1} + R_k,$$

where $t^i = (t, \overset{i\text{-times}}{\dots}, t)$, $F^{(i)}(x) \in \mathcal{L}(\mathbb{R}^n \times \overset{i\text{-times}}{\dots} \times \mathbb{R}^n, \mathbb{R}^n)$ for each $i = 1, 2, \dots$, and

$$\|R_k\| \leq \frac{1}{k!} \sup_{0 < h < 1} \|F^{(k)}(x+ht)\| \|t\|^k.$$

Observe that the term $F^{(i)}(x)t^i \in \mathbb{R}^n$, for each i , is well-defined in view of the Definition.

6.3 Generalized divided difference

You may apply the generalized divided difference for vector-valued functions $F: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ to the case of scalar-valued functions, as stated in Sec. 1.2.9. The operator for divided differences is described as a mapping $F[\cdot, \cdot]: \Omega \times \Omega \rightarrow L(\mathbb{R}^n)$.

$$F[x, y](x - y) = F(x) - F(y), \quad \forall (x, y) \in \Omega^2. \tag{6.4}$$

Moreover, if F is Frechet-differentiable, then the first-order divided difference can be defined as (Alsaedi, A. 2015).

$$F[x+h, x] = \int_0^1 F'(x+th) dt, \quad \forall (x, h) \in \Omega^2. \tag{6.5}$$

However, for the computational purposes, the divided difference operator $F[x, y]$ can be considered as the approximation of Jacobian matrix $F_0(x)$, as the operator $[\cdot, \cdot; F]: \Omega \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^n)$, which is being defined as the divided difference matrix with elements,

$$[x, y; F]_{ij} = \frac{f_i(x_1, \dots, x_j, y_{j+1}, \dots, y_n) - f_i(x_1, \dots, x_{j-1}, y_j, \dots, y_n)}{x_j - y_j}, \quad 1 \leq i, j \leq n, \tag{6.6}$$

Wherein $F = (f_1, f_2, \dots, f_n)^T$, $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$.

7. CONCLUSION

All things considered, a major step forward in computational mathematics has been the development and study of novel iterative methods for numerically solving nonlinear equations.

The goal of these methods is to make the convergence process faster, more stable, and more computationally efficient than the Newton-Raphson and Secant methods, which are traditional approaches. The recently suggested approaches frequently use hybrid systems, adaptive parameters, or higher-order derivatives to improve accuracy without drastically raising computing cost. Theoretical research and numerical tests have proven that these approaches converge quicker with fewer iterations than traditional methods, especially for equations that are sensitive to initial assumptions or derivative assessment, and for which standard methods fail. Convergence features, including convergence order and error bounds, have been studied and verified to be resilient and applicable to a wide variety of nonlinear situations. Applying such sophisticated methods can be especially useful in domains where nonlinear models are common, such as engineering, physics, and the applied sciences.

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