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Unified Evaluation of Definite Integrals Involving the I-Function and Special Functions

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Abstract: The evaluation of definite integrals by generalised special functions is a basic part of mathematical analysis and is important in the analysis of integral transformations and generalised hypergeometric functions. In this study, the unified evaluation of several definite integrals of I-function and generalised hypergeometric functions is discussed. To derive three major integral results under suitable convergence conditions and limits the study combines the Mellin–Barnes-type contour integral representation of the I-function with known definite integral identities and formulae for the hypergeometric function. The analytical method entails using proven definite integral formulae, switching between integration and summation under suitable mathematical circumstances, and substituting integral and series representations. In addition, the main results are specialised by using some suitable parameter changes to get nine corollaries involving the generalised H-function, Fox H-function and G-function. These reductions illustrate the ability to represent different types of integral expressions in one analytical framework, and emphasise the relationships between different classes of generalised special functions. The book contributes to the mathematical research of generalised special functions and their integral representations and provides a systematic way to evaluate particular definite integrals.

Keywords: Definite integrals, I-function, Mellin–Barnes contour integrals, Generalized hypergeometric functions, Fox H-function, H̄-function, G-function.

1. INTRODUCTION

Numerous applications of mathematical analysis are deeply connected with special functions, such as the evaluation of differential equations, mathematical physics, definite integrals, and integral transformations. Generalised hypergeometric functions (Slater, 1966) provide a systematic way of expressing and developing relationships between many of the classical mathematical functions. With the introduction of generalised special functions the analytical ways to evaluate complicated integral expressions have been significantly broadened.

These innovations have gained considerable attention, in particular the G-function and the H-function, which have numerous applications in mathematics. Fox (1961) and Braaksma (1964)

have studied the G- and H-functions as symmetrical Fourier kernels and the asymptotic expansions and analytic continuation of Barnes integrals, respectively. Mathai and Saxena (1978) further developed and extended the concepts of the H-function and its applications to statistics and other areas of mathematics. Later, the series obtained from Feynman integrals were generalised as hypergeometric series by Inayat-Hussain (1987).

Rathie (1997) made another generalization of the Fox H-function, a Mellin-Barnes type contour integral of products of gamma functions raised to appropriate powers, into the I-function. The generalised I-function provides a good mathematical generalisation for a large class of special functions, under appropriate restrictions on its parameters. There has been additional research on its expansion formulae and analytical properties. A study by Satyanarayana et al. (2015) highlighted the importance of this study in the wider context of the theory of generalised special functions.

There has been a lot of interest in studying how generalised special functions may be used to evaluate definite integrals. The analytical identities presented by Qureshi et al. (2011) for a few definite integral formulae might serve as useful references for further mathematical research. Agarwal and Chand (2012) derived finite integral formulae involving generalised Mellin-Barnes-type contour integral and Daiya et al. (2016) dealt with definite integral formulas with multivariable H-function and Srivastava polynomials. The results presented here demonstrate the importance of generalised contour-integral representations for larger classes of integral identities.

This paper unifies definite integral evaluation by using the I-function and generalised hypergeometric functions. Appropriate convergence conditions are considered with the help of the Mellin-Barnes integral representation, well-known definite integral formulas and hypergeometric function identities and three main integral findings appear. Further, nine corollaries involving the generalised H-function, the Fox H-function and the G-function can be derived, under certain parameter restrictions. Hence, it provides an uniform analytical model to represent some definite integrals and to connect important classes of generalised special functions.

2. PRELIMINARIES

Over the course of the last four decays or so, a considerable number of writers have investigated a wide range of fascinating and useful extensions of well-known special

functions. Some examples are Satyanarayana et al. (2015), Saxena (2008), Saxena et al. (2007), Shantha et al. (2012), & Vyas and Rathie (1998). These are only a few instances. In this context, the notation and complete description of the Γ¨-function, which was first developed by Rathie, are presented in terms of the Mellin-Barnes type integral.

$$\bar{I}(z) = \bar{I}_{p,q}^{m,n}\left(z \left| \begin{matrix} {}_1\left(a_j, \alpha_j; A_j'\right)_n, & {}_{n+1}\left(a_j, \alpha_j; A_j''\right)_p \\ {}_1\left(b_j, \beta_j; B_j''\right)_m, & {}_{m+1}\left(b_j, \beta_j; B_j'\right)_q \end{matrix} \right.\right) = \frac{1}{2\pi\omega}\int_L \chi(s) z^s ds, \tag{1}$$

For any non-zero value of z and

$$\chi(s) = \frac{\prod_{j=1}^{m}\Gamma^{B_j''}\left(b_j - \beta_j s\right)\prod_{j=1}^{n}\Gamma^{A_j'}\left(1 - a_j + \alpha_j s\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j''}\left(a_j - \alpha_j s\right)\prod_{j=m+1}^{q}\Gamma^{B_j'}\left(1 - b_j + \beta_j s\right)}, \tag{2}$$

In the case when $\Delta > 0$, the integral (1) comes to an end when the absolute value of arg z is less than half of $\Delta\pi$.

$$\Delta = \sum_{j=1}^{m} B_j''\beta_j - \sum_{j=m+1}^{q} B_j'\beta_j + \sum_{j=1}^{n} A_j'\alpha_j - \sum_{j=n+1}^{p} A_j''\alpha_j \tag{3}$$

In a recent publication by Qureshi et al. (2011), the following formulae were defined.

$$\int_0^\infty \left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-u-1} dx = \frac{\sqrt{\pi}}{2\rho\left(4\rho\tau + \gamma\right)^{u+1/2}}\frac{\Gamma\left(u + 1/2\right)}{\Gamma\left(u + 1\right)}, \tag{4}$$

$$\left(\rho > 0;\ \tau \geq 0;\ \gamma + 4\rho\tau > 0;\ \left(\Re(u) + 1/2\right) > 0\right).$$

$$\int_0^\infty \frac{1}{x^2}\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-u-1} dx = \frac{\sqrt{\pi}}{2\tau\left(4\rho\tau + \gamma\right)^{u+1/2}}\frac{\Gamma\left(u + 1/2\right)}{\Gamma\left(u + 1\right)}, \tag{5}$$

$$\left(\rho \geq 0;\ \tau > 0;\ \gamma + 4\rho\tau > 0;\ \left(\Re(u) + 1/2\right) > 0\right).$$

$$\int_0^\infty \left(\rho + \frac{\tau}{x^2}\right)\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-u-1} dx = \frac{\sqrt{\pi}}{\left(4\rho\tau + \gamma\right)^{u+1/2}}\frac{\Gamma\left(u + 1/2\right)}{\Gamma\left(u + 1\right)}, \tag{6}$$

$$(\rho > 0;\ \tau > 0;\ \gamma + 4\rho\tau > 0;\ (\Re(u) + 1/2) > 0).$$

We will need to consult Slater (1966) for the following formulae that are relevant to our inquiry.

$$ {}_2F_1\left(a, b, c+\frac{1}{2}; x\right)\ {}_2F_1\left(c-a, c-b, c+\frac{1}{2}; x\right) = \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c+\frac{1}{2}\right)_r} a_r x^r. \tag{7}$$

3. RESULTS

Theorem 1. If $\rho > 0; \tau \geq 0; \gamma + 4\rho\tau > 0; (\Re(\lambda) + 1/2) > 0, -\frac{1}{2} < (a - b - c) < \frac{1}{2}; |\arg z| < \frac{1}{2}\Delta,$ If equation (3) is satisfied and $\Delta > 0$, therefore the following formula is valid:

$$\begin{aligned}
&\int_0^{\infty} \left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} {}_2F_1\left(a, b, c+\frac{1}{2}; y\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right) \\
&\quad \times {}_2F_1\left(c-a, c-b, c+\frac{1}{2}; y\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right) \bar{I}_{p,q}^{m,n}\left[z\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega}\right] dx \\
&\quad = \frac{\sqrt{\pi}}{2\rho\,(4\rho\tau+\gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c+\frac{1}{2}\right)_r} \frac{a_r\, y^r}{(4\rho\tau+\gamma)^{\mu r}} \\
&\quad \times \bar{I}_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}} \middle| \begin{matrix} \left(\frac{1}{2}-\lambda-\mu r, \omega;\ 1\right), {}_1\left(a_j, \alpha_i;\ A_j'\right)_n, {}_{n+1}\left(a_j, \alpha_i;\ A_j''\right)_p \\ {}_1\left(b_j, \beta_j;\ B_j''\right)_m, {}_{m+1}\left(b_j, \beta_j;\ B_j'\right)_q, (-\lambda-\mu r, \omega;\ 1) \end{matrix}\right].
\end{aligned} \tag{8}$$

Proof. We may deduce this from (1), (4), and (7):

$$= \int_0^{\infty} \left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c+\frac{1}{2}\right)_r} a_r y^r \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu r} \times \frac{1}{(2\pi\omega)} \int_L \chi(s)\,(z)^s \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega s} ds dx \tag{9}$$

Changing the sequence of integration & summation, given the right circumstances, yields

$$= \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c+\frac{1}{2}\right)_r} a_r\, y^r \frac{1}{(2\pi\,\omega)} \int_L \chi(s)\,(z)^s \left\{\int_0^{\infty} \left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-\mu r-\omega s-1} dx\right\} ds, \tag{10}$$

The integral given above is transformed using equation (4).

$$= \frac{\sqrt{\pi}}{2\rho\,(4\rho\tau+\gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c+\frac{1}{2}\right)_r} \frac{a_r\, y^r}{(4\rho\tau+\gamma)^{\mu r}} \frac{1}{(2\pi\,\omega)} \int_L \chi(s) \left(\frac{z}{(4\rho\tau+\gamma)^{\omega}}\right)^s \frac{\Gamma\left(\lambda+\mu r+\omega s+1/2\right)}{\Gamma\left(\lambda+\mu r+\omega s+1\right)} ds,$$

By combining the previous equation with equation (1), we get to (8). The first theorem's proof is now complete.

Theorem 2. If $\rho \geq 0;\ \tau > 0;\ \gamma + 4\rho\tau > 0;\ (\Re(\lambda) + 1/2) > 0\ , -\frac{1}{2} < (a - b - c) < \frac{1}{2};\ |\arg z| < \frac{1}{2}\Delta,$

In cases when Δ is more than zero and is provided by equation (3), the following formula is valid:

$$\int_0^\infty \frac{1}{x^2}\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} {}_2F_1\left(a, b, c + \frac{1}{2}; y\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right)$$

$$\times {}_2F_1\left(c - a, c - b, c + \frac{1}{2}; y\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right) \bar{I}_{p,q}^{m,n}\left[z\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega}\right] dx$$

$$= \frac{\sqrt{\pi}}{2\tau(4\rho\tau + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} \frac{a_r y^r}{(4\rho\tau + \gamma)^{\mu r}}$$

$$\times \bar{I}_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau + \gamma)^{\omega}} \middle| \begin{matrix} \left(\frac{1}{2} - \lambda - \mu r, \omega; 1\right), {}_1\left(a_j, \alpha_i; A_j'\right)_n, {}_{n+1}\left(a_j, \alpha_i; A_j''\right)_p \\ {}_1\left(b_j, \beta_j; B_j''\right)_m, {}_{m+1}\left(b_j, \beta_j; B_j'\right)_q, (-\lambda - \mu r, \omega; 1) \end{matrix}\right]. \tag{11}$$

Proof. Thanks to (1), (5), and (7), we can deduce

$$= \int_0^\infty \frac{1}{x^2}\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} a_r y^r \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu r}$$
$$\times \frac{1}{(2\pi\omega)} \int_L \chi(s)\,(z)^s \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega s} ds dx, \tag{12}$$

Switching the sequence of integration and summation yields

$$= \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} a_r y^r \frac{1}{(2\pi\omega)} \int_L \chi(s)\,(z)^s \left\{\int_0^\infty \frac{1}{x^2}\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-\mu r-\omega s-1} dx\right\} ds, \tag{13}$$

Equation (5) allows us to transform the integral given above into

$$= \frac{\sqrt{\pi}}{2\tau\,(4\rho\tau + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} \frac{a_r\, y^r}{(4\rho\tau + \gamma)^{\mu r}} \frac{1}{(2\pi\omega)} \int_L \chi(s) \left(\frac{z}{(4\rho\tau + \gamma)^{\omega}}\right)^s \frac{\Gamma(\lambda + \mu r + \omega s + 1/2)}{\Gamma(\lambda + \mu r + \omega s + 1)} ds$$

We get at (11) by plugging the previous equation into the context of equation (1). The proof of Theorem 2 is now complete.

Theorem 3. If $|\rho > 0;\ \tau > 0;\ \gamma + 4\rho\tau > 0;\ (\Re(\lambda) + 1/2) > 0\ , -\frac{1}{2} < (a - b - c) < \frac{1}{2};\ |\arg z| < \frac{1}{2}\Delta,$

If equation (3) is satisfied and $\Delta > 0$, therefore the following formula is valid:

$$\int_0^\infty \left(\rho + \frac{\tau}{x^2}\right)\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} {}_2F_1\left(a, b, c + \frac{1}{2}; y\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right)$$
$$\times {}_2F_1\left(c - a, c - b, c + \frac{1}{2}; y\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right) \bar{I}_{p,q}^{m,n}\left[z\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega}\right] dx$$
$$= \frac{\sqrt{\pi}}{(4\rho\tau + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} \frac{a_r y^r}{(4\rho\tau + \gamma)^{\mu r}}$$

$$\times \bar{I}_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau + \gamma)^{\omega}} \middle| \begin{matrix} \left(\frac{1}{2} - \lambda - \mu r, \omega; 1\right), {}_1\left(a_j, \alpha_i; A_j'\right)_n, {}_{n+1}\left(a_j, \alpha_i; A_j''\right)_p \\ {}_1\left(b_j, \beta_j; B_j''\right)_m, {}_{m+1}\left(b_j, \beta_j; B_j'\right)_q, (-\lambda - \mu r, \omega; 1) \end{matrix}\right] \quad (14)$$

Proof. Equations (1), (6), and (7) allow us to conclude that

$$= \int_0^\infty \left(\rho + \frac{\tau}{x^2}\right)\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} a_r y^r \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu r}$$
$$\times \frac{1}{(2\pi\omega)} \int_L \chi(s)\,(z)^s \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega s} ds dx, \quad (15)$$

Under the correct assumption, we get by switching the sequence of integration and summation.

$$= \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} a_r y^r \frac{1}{(2\pi\omega)} \int_L \chi(s)\,(z)^s \left\{\int_0^\infty \left(\rho + \frac{\tau}{x^2}\right)\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-\mu r-\omega s-1} dx\right\} ds, \quad (16)$$

Equation (6) allows us to transform the integral given above into

$$= \frac{\sqrt{\pi}}{(4\rho\tau + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} \frac{a_r y^r}{(4\rho\tau + \gamma)^{\mu r}} \frac{1}{(2\pi\omega)} \int_L \chi(s) \left(\frac{z}{(4\rho\tau + \gamma)^{\omega}}\right)^s \frac{\Gamma(\lambda + \mu r + \omega s + 1/2)}{\Gamma(\lambda + \mu r + \omega s + 1)} ds,$$

With (1) in mind, we can use the previous equation to get to (14) now. The third theorem's proof is now complete.

In Corollaries (17) to (19), (20) to (22) and (23) to (25) correspondingly, we find some known H^{-}-function, H-function, and G-function, from which we derive new integral formulas. Part

I. The three corollaries that result from using the same approach to obtaining Theorems 1 to 3 are known as the H¯-function, and they were developed by Inayat-Hussain in 1987. By plugging these values into equation (1), we get

$$\bar{I}_{p,q}^{m,n}\left(z\left|\begin{matrix}{}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;A_j''\right)_p\\ {}_1\left(b_j,\beta_j;B_j''\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right.\right)=\bar{H}_{p,q}^{m,n}\left(z\left|\begin{matrix}{}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;1\right)_p\\ {}_1\left(b_j,\beta_j;1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right.\right)$$

Now, using the aforementioned identity, we may simplify each of Theorems 1-3 as follows:

Corollary 1. We can prove that given that Theorem 1 holds if and only if

$$\begin{aligned}\int_0^\infty \Phi^{-\lambda-1}\,{}_2F_1\left(a,b,c+\frac{1}{2};y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,c-b,c+\frac{1}{2};y\{\Phi\}^{-\mu}\right)\,\bar{H}_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx\\ =\frac{\sqrt{\pi}}{2\rho\left(4\rho\tau+\gamma\right)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}\\ \times\bar{H}_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;\ 1\right),{}_1\left(a_j,\alpha_i;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_i;\ 1\right)_p\\ {}_1\left(b_j,\beta_j;\ 1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q,\left(-\lambda-\mu r,\omega;\ 1\right)\end{matrix}\right.\right],\end{aligned}\tag{17}$$

where $\Phi=\left[\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right]$.

Corollary 2. Let the condition of Thm. 2. we can be satisfied,

$$\begin{aligned}\int_0^\infty \frac{1}{x^2}\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,b,c+\frac{1}{2};y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,c-b,c+\frac{1}{2};y\{\Phi\}^{-\mu}\right)\,\bar{H}_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx\\ =\frac{\sqrt{\pi}}{2\tau\left(4\rho\tau+\gamma\right)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}\\ \times\bar{H}_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\ \omega;\ 1\right),{}_1\left(a_j,\alpha_i;\ A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_i;\ 1\right)_p\\ {}_1\left(b_j,\beta_j;\ 1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;\ B_j'\right)_q,\left(-\lambda-\mu r,\ \omega;\ 1\right)\end{matrix}\right.\right].\end{aligned}\tag{18}$$

Corollary 3. If the condition in Theorem 3 is met, we get

$$\int_0^\infty \left(\rho+\frac{\tau}{x^2}\right)\ \Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\ \bar{H}_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]\,dx$$
$$=\frac{\sqrt{\pi}}{(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}$$
$$\times\ \bar{H}_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;\,1\right),\,{}_1\left(a_j,\alpha_i;\,A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_i;\,1\right)_p\\ {}_1\left(b_j,\beta_j;\,1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;\,B_j'\right)_q,\left(-\lambda-\mu r,\omega;\,1\right)\end{matrix}\right]. \tag{19}$$

(II). If we use $A_j''=B_j''=A_j'=B_j'=1$ In the set of theorems known as $\bar{I}$-function, which is defined by Fox 1961, Braaksma 1964, and Mathai and Saxena 1978, the set of related functions is reduced to the well-known H-function.

$$\bar{I}_{p,\,q}^{m,\,n}\left(z\,\middle|\,\begin{matrix}{}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;A_j''\right)_p\\ {}_1\left(b_j,\beta_j;B_j''\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right)=H_{p,\,q}^{m,\,n}\left(z\,\middle|\,\begin{matrix}{}_1\left(a_j,\alpha_j;1\right)_p\\ {}_1\left(b_j,\beta_j;1\right)_q\end{matrix}\right)$$

Corollary 4. We can prove that given that Theorem 1 holds if and only if

$$\int_0^\infty \Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\ H_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]\,dx$$
$$=\frac{\sqrt{\pi}}{2\rho\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}H_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;\,1\right),\,{}_1\left(a_j,\alpha_i;\,1\right)_p\\ {}_1\left(b_j,\beta_j;\,1\right)_q,\ \left(-\lambda-\mu r,\omega;\,1\right)\end{matrix}\right]. \tag{20}$$

Corollary 5. Consider the stipulation of Theorem 2. Be content, we acquire.

$$\int_0^\infty \frac{1}{x^2}\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\ H_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]\,dx$$
$$=\frac{\sqrt{\pi}}{2\tau\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}H_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\ \omega;\,1\right),\,{}_1\left(a_j,\alpha_i;\,1\right)_p\\ {}_1\left(b_j,\beta_j;\,1\right)_q,\ \left(-\lambda-\mu r,\ \omega;\,1\right)\end{matrix}\right]. \tag{21}$$

Corollary 6. If the condition in Theorem 3 is met, we get

$$\int_0^\infty \left(\rho+\frac{\tau}{x^2}\right)\ \Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\ H_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]\,dx$$
$$=\frac{\sqrt{\pi}}{(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}H_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;\,1\right),\,{}_1\left(a_j,\alpha_i;\,1\right)_p\\ {}_1\left(b_j,\beta_j;\,1\right)_q,\ \left(-\lambda-\mu r,\omega;\,1\right)\end{matrix}\right]. \tag{22}$$

(III). If we set $A_j'' = B_j'' = A_j' = B_j' = 1$ and $\alpha_j = \beta_j = 1$ The I¯-function is reduced to the G-function in Theorems 1 to 3.

$$\bar{I}_{p,q}^{m,n}\left(z\left|\begin{matrix}{}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;A_j''\right)_p\\ {}_1\left(b_j,\beta_j;B_j''\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right.\right)=G_{p,q}^{m,n}\left(z\left|\begin{matrix}{}_1\left(a_j,\ 1;\ 1\right)_p\\ {}_1\left(b_j,1;\ 1\right)_q\end{matrix}\right.\right)$$

Corollary 7. If the condition in Theorem 1 is met, we have

$$\int_0^\infty \Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,G_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]\,dx$$
$$=\frac{\sqrt{\pi}}{2\rho\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}G_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;\ 1\right),\,{}_1\left(a_j,1;\,1\right)_p\\ {}_1\left(b_j,1;\ 1\right)_q,\ \ (-\lambda-\mu r,\omega;\ 1)\end{matrix}\right.\right]. \tag{23}$$

Corollary 8. Let's look at the condition of Theorem 2. be happy with what we get

$$\int_0^\infty \frac{1}{x^2}\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,G_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]\,dx$$
$$=\frac{\sqrt{\pi}}{2\tau\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}G_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;\ 1\right),\,{}_1\left(a_j,1;\,1\right)_p\\ {}_1\left(b_j,1;\ 1\right)_q,\ \ (-\lambda-\mu r,\omega;\ 1)\end{matrix}\right.\right] \tag{24}$$

Corollary 9. If the condition in Theorem 3 is met, we get

$$\int_0^\infty \left(\rho+\frac{\tau}{x^2}\right)\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,G_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]\,dx$$
$$=\frac{\sqrt{\pi}}{(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu r}}G_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;\ 1\right),\,{}_1\left(a_j,1;\,1\right)_p\\ {}_1\left(b_j,1;\ 1\right)_q,\ \ (-\lambda-\mu r,\omega;\ 1)\end{matrix}\right.\right]. \tag{25}$$

4. CONCLUSION

Using Mellin-Barnes type contour integral representations, the I-function, and generalised hypergeometric functions, this study has sought to determine the unified evaluation of various definite integrals. There are three main integral formulae proposed under the right convergence conditions and the limits of the parameters, based on known definite integral identities and hypergeometric function relationships. Generalised integral representations are shown to be able to represent a complex definite integral in terms of special functions in the analytical technique. nine additional findings have been deduced from the primary findings

using appropriate parameter specialisation. These corollaries are for the generalised H-function, Fox H-function and G-function, and indicate the connections between various kinds of special functions. The findings provide a common mathematical structure to the selected integral identities and also the particular cases that represent these identities. The study proves that Mellin-Barnes contour integral methods are important in generalised special function theory and that the I-function is helpful for analytically treating definite integrals. The presented framework could support additional research on related integral transformations, fractional calculus and other generalised special functions.

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